Conjugate gradient

We now have:

Start: x_0 , $r_0 = Ax_0 - b$, $p_0 = -r_0$

Step:

$$x_{k+1} = x_k + \alpha_k p_k$$

$$\alpha_k = \frac{-r_k' A p_k}{p_k' A p_k}$$

$$p_{k+1} = -r_{k+1} + \beta_{k+1} p_k$$

$$\beta_{k+1} = \frac{p_k' A r_{k+1}}{p_k' A p_k}$$

We can make this more efficient

Again this gives

$$\frac{1}{2}\left[(x_k + \alpha_k p_k)' A (x_k + \alpha_k p_k) \right]$$
$$- b_k \ (x_k + \alpha_k p_k)$$

min is at:

$$\frac{-(A x_k - b)' p_k}{p_k' A p_k}$$

write

$$r_k = A x_k - b$$

so $$\alpha_k = \frac{- r_k' \, p_k}{p_k' A p_k}$$

Conjugate direction methods:

- a set of vectors, $p_0 \cdots p_n$ is conjugate for A positive definite if

$$p_i' A p_j = 0 \qquad i \neq j$$

- Assume we wish to <u>min</u>

$$\frac{x' A x}{2} - b' x$$

- useful because:

  a) solution to $Ax = b$ for A p.d.

  b) $\min_x \|Ux - b\|^2$ is like this

- now write

$$x = \alpha_0 p_0 + \alpha_1 p_1 + \cdots$$

Conjugate direction in incremental form

Start with x_0, p_0

$$x_1 = x_0 + \alpha_0 p_0$$

now min wrt α_0

to get

$$\frac{(Ax_0 - b)' p_0}{p_0' A p_0} = \alpha_0$$

write

$$r_k = (Ax_k - b)$$

and get

$$x_{k+1} = x_k + \alpha_k p_k$$

$$\alpha_k = \frac{r_k' p_k}{p_k' A p_k}$$

$$r_{k+1} = r_k + \alpha_k A p_k$$

gate gradient (Simple form)

Start: x_0, $r_0 = Ax_0 - b$, $p_0 = -r_0$

Step:

$$x_{k+1} = x_k + \alpha_k p_k$$

$$\alpha_k = \frac{-r_k' p_k}{p_k' A p_k}$$

$$p_{k+1} = -r_{k+1} + \beta_{k+1} p_k$$

$$\beta_{k+1} = \frac{r_{k+1}' A p_k}{p_k^T A p_k}$$

$$r_{k+1} = r_k + \alpha_k A p_k$$

Properties of conj. direction

$$r_k' p_i = 0, \quad \forall i < k$$

(Show this by induction)

$$r_k' r_i = 0 \quad \forall i < k$$

(thm 5.3 at end).

1A

Conjugate gradient.

<u>Cleaner form:</u>

- By properties, we have

$$\alpha_{K\cancel{+1}} = \frac{r_K' r_K}{p_K' A p_K}$$

- Now $\alpha_K A p_K = r_{K+1} - r_K$

So
$$\beta_{K+1} = r_{K+1}' \frac{(r_{K+1} - r_K)}{\alpha_K} \cdot \frac{1}{p_K' A p_K}$$

$$= \frac{r_{K+1}' (r_{K+1} - r_K)}{r_K' r_K}$$

$$= \frac{r_{K+1}' r_{K+1}}{r_K' r_K}$$ (By properties)