One solution:

$$\min_{x,z} \ \frac{1}{2}|Ax-b|^2 + \alpha|z|_1 \qquad \text{s.t} \quad x = z$$

with $\underbrace{\text{ALM} + \text{Splitting}}$ → class of methods known as ADMM

alternating directions method of multipliers

x^{k+1}: $$\operatorname{argmin} \ \frac{1}{2}(Ax-b)^T(Ax-b) + \lambda^T(x - z^k) + \frac{\rho^k}{2}(x-z^k)^T(x-z^k).$$

so solve:

$$\left(A^TA + \rho^k I\right)x = A^Tb + \rho^k z^k$$

z^{k+1} is more interesting:

$$\operatorname{argmin} \sum_i \left[\lambda_i(x_i^{k+1} - z_i) + \frac{\rho^k}{2}(x_i^{k+1} - z_i)^2 + \alpha|z_i|\right]$$

Notice

sum of terms, one in each z_i.

now consider the model problem

$$\min_z \frac{1}{2}(z-u)^2 + \alpha|z| \qquad \text{in one var} \quad \alpha > 0$$

a) assume $z > 0$:

$$z - u + \alpha = 0, \qquad z = u - \alpha$$

b) assume $z < 0$:

$$z - u - \alpha = 0, \qquad z = u + \alpha$$

so we know min for $u > \alpha$, $u < -\alpha$.

But what about $-\alpha < u < \alpha$?

I claim $z = 0$

proof cost for $z = 0$ is $\frac{1}{2}u^2$

now consider $z \neq 0$ . .

cost is $\frac{1}{2}(z-u)^2 + \alpha|z|$

and show $\frac{1}{2}(z-u)^2 + \alpha|z| - \frac{1}{2}u^2 > 0$!

our problem is:

$$\min_z \quad \lambda(x-z) + \frac{\rho}{2}(x-z)^2 + \alpha|z|$$

fixed x, λ, ρ.

which is

$$\frac{\rho}{2}\left(z - \left(x + \frac{\lambda}{\rho}\right)\right)^2 + g(x,\lambda) + \alpha|z|$$

so we solve by

$$z_i = \begin{cases} x_i^{k+1} + \frac{\lambda_i^k}{\rho^k} - \alpha & \text{if } x_i^{k+1} + \frac{\lambda_i^k}{\rho^k} > \alpha. \\ 0 & \text{otherwise} \\ x_i^{k+1} + \frac{\lambda_i^k}{\rho^k} + \alpha & \text{if } \quad " \quad < -\alpha \end{cases}$$

This is known as "Shrinking" or "Shrinkage"

Application:

noisy image I^*

we would like to denoise.

One approach: Total variation denoising

choose I that minimizes

$$|I - I^*|^2 + \alpha \left[\left| \frac{\partial I}{\partial x} \right|_1 + \left| \frac{\partial I}{\partial y} \right|_1 \right]$$

~~with~~ arrange image into vector I

notice there is a matrix G such that

GI is an estimate of $\begin{bmatrix} \frac{\partial I}{\partial x} \\ \frac{\partial I}{\partial y} \end{bmatrix}$

G is big but sparse.

we could do:

$$\min_{I} \; |I - I^*|^2 + \alpha |z|_1$$

such that $GI = z$

notice that there is more than one I term in the equation for z.
If this disturbs you, can do:

$$\min_{I} \; |I - I^*|^2 + \alpha |z|$$

s.t. $GI = g$

$g = z$.

then solve $I^k, g^k, z^k, \lambda^k, p^k$