Linear programs

$$\max \; c^T x$$
$$\text{st} \quad A\underline{x} = \underline{b}$$
$$M\underline{x} \leq \underline{n}$$

Notice: ~~cases~~:

- Domain is always convex
  (i.e. if x_1, x_2 are in, then $tx_1 + (1-t)x_2$ is in for $0 \leq t \leq 1$)
- called the Feasible Set
- could be
  - empty
  - compact (then its a polytope)
  - a cone
- solution could be
  - non existent (empty feasible set)
  - a k-face of the feasible set (k usually zero)
  - infinite

Linear programs

we can always convert to
Standard form (equational form).

$$\max\ c^T x$$
$$\text{s.t.}\quad Ax = b$$
$$x \geq 0$$

← Not the same A, b – sorry!

By:

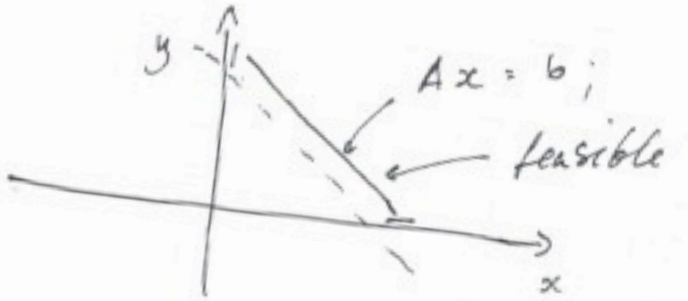
- introduce a slack for each inequality, mult by -1 if necessary.
- if we have x_i which is unbounded, then replace w/ $x_u - x_v$, $x_u \geq 0$, $x_v \geq 0$

The Simplex method:

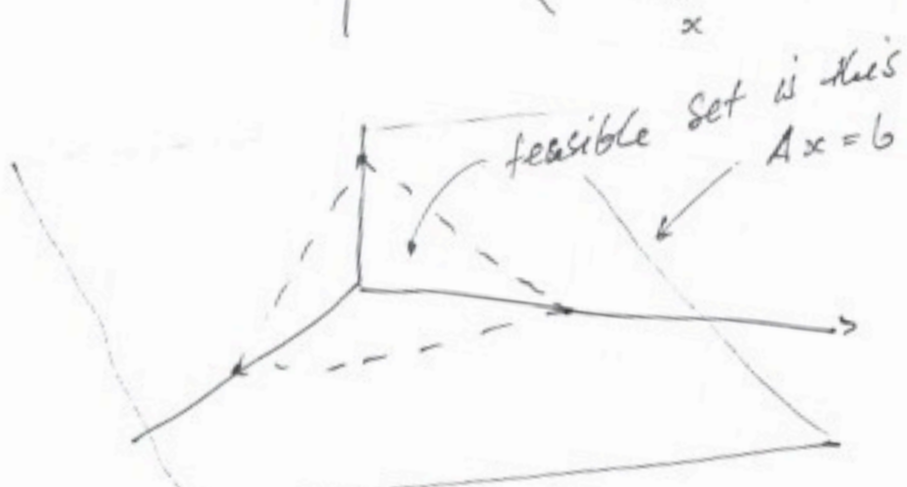
- Assume there is at least one x st
  $Ax = b$
- rows of A are linearly indep.

Notice geom of standard form
- take +ve orthant, slice w/ linear space

2D:



3D:



All this means that there must be
<u>basic</u> solns.

. where "enough" of the vars are zero

$$m \updownarrow \overset{\xleftrightarrow{\;n\;}}{A} \quad x = b$$

. basic soln.
  . m-element set of <u>indices</u> B st.
    $A_B = \{$cols of A indexed by $B\}$
    has full rank.
  . $x_j = 0 \;, j \notin B$.

. Notice if we know B, we know $\underline{x}$ & (nonsingular A_B)

. if there is a feasible ~~soln~~ point, ~~with~~ and ~~bound~~ objective is bounded above, then there is an optimal soln

. If there is an optimal soln, then there is a basic optimal soln

Simplex example
(after Matoušek + Gartner)

$$\max \quad x_1 + x_2$$

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 \quad &\leq 3 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

to get standard form, introduce slacks x_3, x_4, x_5

$$\max \quad x_1 + x_2$$

$$\begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ \text{all } x &\geq 0 \end{aligned}$$

represent with Tableau:

$$x_3 = 1 + x_1 - x_2$$
$$x_4 = 3 - x_1$$
$$x_5 = 2 - x_2$$

interpret as $(0, 0, 1, 3, 2)$.

Notice: (a) - feasible

Q: can we improve?

$z = x_1 + x_2$. currently, 0

~~Notice x_1~~ Strategy: choose a variable that is 0 and one that is non-zero and swap:

in this case, notice that
- x_2 can get bigger
- and this would make z bigger

$\Rightarrow$ swap x_3, x_2

New Tableau :

$$x_2 = 1 + x_1 - x_3$$
$$x_4 = 3 - x_1$$
$$x_5 = 2 - x_2 = 2 - (1 + x_1 - x_3) = 1 - x_1 + x_3$$

interpret as $(0, 1, 0, 3, 1)$

and $z = x_1 + x_2 = x_1 + (1 + x_1 - x_3)$
$= 1 + 2x_1 - x_3$

Value is 1 (so it got better)

Notice: · to increase z, we need to increase x_1

· but x_1 can't get bigger than 1 (cause $x_5 = 1 - x_1 + x_3 \geq 0$

· x_1 goes in, x_5 goes out.

· subs $x_1 = 1 + x_3 - x_5$ to get

$$\left.\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \end{aligned}\right\} \rightsquigarrow (1, 2, 0, 2, 0)$$

$$z = 3 + x_3 - 2x_5$$

<u>Notice</u>

- increasing x_3 will inc. z
- it can go to 2

$\Rightarrow$ x_3 goes in, x_4 goes out.

$$\left.\begin{aligned} x_1 &= 3 - x_4 \\ x_2 &= 2 - x_5 \\ x_3 &= 2 - x_4 + x_5 \end{aligned}\right] \to (3, 2, 2, 0, 0)$$

$$z = 5 - x_4 - x_5$$

$(x_i \geq 0)$

- No move makes z better

so <u>stop</u>

Notice we are moving along 1-faces of feasible polytope.

i.e. tableau = $\underbrace{\text{0-face}}_{\text{vertex}}$

1 eqn goes out, another goes in = move to another 0-face connected to last 0-face by a 1-face.

- and we do this along edges such that $c \cdot (\text{edge vector}) > 0$

Killer algorithmic question:

- who goes in, who goes out?

This turns out to be complicated as methods can cycle.

Cycling

- all available 0-verts might have same value of objective.
- if this happens, we could come back to where we came from.

Various rules for pivoting:

- largest coefficient in objective
- largest increase in objective
- Steepest edge

$$\frac{c^T (x_{new} - x_{old})}{\| x_{new} - x_{old} \|} \quad \text{is best}$$

V. good: approx this test.

- random
  (best provable bounds)
- Bland

No cycling | . incoming var has smallest index
| . ditto outgoing, if choice

Very painful $\left\{\begin{matrix}\text{practical}\\ \text{intellectual}\end{matrix}\right\}$ problem (10)

→ Simplex is super-good on large problems

→ no pivot rule known can be proved polynomial

→ existence of a polynomial pivot rule is a <u>major</u> open qu. in geometry

One can prove various average case results (Spielman + Teng, etc).