Interior point methods

problem $\min f(x)$

s.t. $g_i(x) \leq 0$

assume that feasible set is not empty

Strategy: avoid the boundary, cause progress on the boundary may be slow

- hard to take large steps, cause active constraints might change
- overstep: - recovering might be hard

Instead, introduce a barrier function,

- ∞ at boundary
- large close to boundary

then solve; make barrier closer to boundary repeat.

Generally, we engage only w/ convex ③
feasible sets

- More interesting than you think
- CURRENTLY, only linear constraints
  (but I'll show others soon!)

Barrier

$$\phi(x) = -\sum_{i=1}^{M} \log(-g_i(x))$$

(Signs depend on direction of ineq.!)

Notice: $g_i(x) < 0$, ϕ is finite ~~and~~

$g_i(x) = 0$ ϕ is $+\infty$

$g_i(x)$ very neg ϕ_i is small ~~[illegible]~~

Now consider

$$\min_x \; f(x) + \frac{1}{t} \varphi(x)$$

at small t, soln is far in interior
big t, soln can approach boundary

Usually : phrase as

$$\min_x \; t f(x) + \varphi(x)$$

It is usual to have linear constraints
as well, so

$$\min_x \; t f(x) + \varphi(x) \quad \text{st} \quad Ax = b$$

<u>Notice</u>:

f convex, g_i convex means

$tf + \varphi$ is convex.

Now consider $x^*(t)$ which is <u>soln</u>

$$A x^* = b$$

$$g_i(x^*) < 0$$

(cause it is soln!)

$$\nabla_x \mathcal{L} = t \nabla_x f(x^*) + \nabla_x \varphi(x^*) + \nu^T A = 0$$

grad of lagrangian is zero at soln

$$\nabla_x \mathcal{L} = 0 = t \nabla_x f(x^*) + \sum_i \left[\frac{-1}{g_i(x^*)} \nabla_x g_i(x^*) \right] + \nu^T A = 0$$

So:

$$\nabla_x f(x^*) + \sum_i \left[\frac{-1}{t\, g_i(x^*)}\right] \nabla_x g_i(x^*) + \frac{\nu}{t}^T A = 0$$

write:

$$\lambda_i^* = \frac{-1}{t\, g_i(x^*)} \quad , \quad \nu^* = \frac{\nu}{t}$$

then

$$\nabla_x f(x^*) + \sum_i \lambda_i^* \nabla_x g_i(x^*) + \nu^{*T} A = 0$$

$$\lambda_i^* \geq 0$$

but this is dual feasibility for the original problem, cause x^* is feasible

Furthermore, we have that

x^* minimizes

$$\mathcal{L}(x, \lambda^*, \nu^*) = f + \sum_i \lambda_i^* g_i + \nu^{*T}(Ax - b)$$

So: the dual is ~~[illegible]~~ not infinite at x^* [(λ^*, ν^*) are dual feasible]

Further, we can evaluate the dual at (λ^*, ν^*)

$$\begin{aligned} g(\lambda^*, \nu^*) &= \text{dual} \\ &= \inf_x \mathcal{L}(x, \lambda^*, \nu^*) \\ &= \mathcal{L}(x^*, \lambda^*, \nu^*) \quad [\text{cause } x^* \text{ minimizes}] \\ &= f(x^*) + \underbrace{\sum_{i=1}^m \left[\frac{-1}{t\, g_i(x^*)}\right] g_i(x^*)}_{-\frac{m}{t}} + \underbrace{\nu^{*T}[Ax - b]}_{0} \end{aligned}$$

<u>So</u>:

- primal has value $f(x^*)$
- dual has value $f(x^*) - \frac{m}{t}$

- <u>BUT</u> dual is lower bound on value of the problem ($g(\lambda, \nu) \leq f(x)$).
- so soln is trapped in range $[f(x^*) - \frac{m}{t}, f(x^*)]$

<u>Algorithm</u>:

- Start with x a feasible point, t small ($t=1$, say). $\mu > 1$, $\varepsilon > 0$

- Solve
$$\min_x \; t f(x) + \varphi(x)$$
$$\underline{st} \quad Ax - b = 0$$
(using, say ALM; ADMM; etc)

- if $\frac{m}{t} < \varepsilon$ quit, else $t = \mu t$

Now look at problem

$$\min_x \quad t f(x) + \varphi(x)$$
$$\text{st } Ax = b$$

When we solve, get x^*, λ_i^*, ν^*

and

$$Ax - b = 0$$
$$g_i(x) \leq 0$$
$$\lambda_i^* \geq 0$$
$$\nabla f + \sum_i \lambda_i^* \nabla g_i + \nu^{*T} A = 0$$
$$-\lambda_i^* g_i = \frac{1}{t}$$

here, we got λ^*, ν^* from x^* — what if they are vars!

KKT for original problem

$$Ax - b = 0$$
$$g_i(x) \leq 0$$
$$\lambda_i \geq 0$$
$$\nabla f + \sum_i \lambda_i \nabla g_i + \nu^T A = 0$$
$$\lambda_i g_i = 0$$