A new kind of convex constraint

for $F$ a matrix, write

$F \succ 0 \iff F$ is the

\textit{positive definite}

$\iff x^T F x > 0$, for ANY $x$.

this constraint yields a convex set of

matrices.

$F_x \succ 0$, $F_\beta \succ 0$

$\Rightarrow (t F_x + (1-t) F_\beta) \succ 0$

$t \in [0, 1]$

[check this – it’s easy]
The problem:

\[ \min_x \quad c^T x \]

\[ \text{st} \quad F_0 + \sum_i x_i F_i \preceq 0 \]

is a **semidefinite program**

**linear programs are SOP's:**

\[ \min \quad e^T u \]

\[ \text{st} \quad M u = n \]

\[ \rho u + q \preceq 0 \]

Transform: \( u = v + H x \)

\[ \min_c \quad c^T x \]

\[ \text{st.} \quad A x + b \succeq 0 \]

\( \rho \) to eliminate equality constraints
But

\[ \text{diag}(v) \preceq 0 \iff v \succeq 0 \]

\[ Ax + b \succeq 0 \iff \text{diag}[Ax + b] \succeq 0 \]

and \[ A = [a_1, \ldots, a_r] \] cols.

and

\[ \text{diag}[Ax + b] \succeq 0 \]

\[ \| \text{diag}(b) + \sum_i x_i \text{diag}(a_i) \| \preceq 0 \]

so

\[ \min c^T x \]

\[ \text{subject to } \text{diag}(b) + \sum_i x_i \text{diag}(a_i) \preceq 0 \]

Which is an SDP
SDP's take a variety of forms

For matrices $A, X$ write

$$\langle A, X \rangle = \text{Trace} [A^T B] = \sum_{ij} A_{ij} B_{ij}$$

Check this equivalence - it's useful and you'll see it often.

SDP's can be written as

$$\min_X \langle C, X \rangle$$

s.t. $$\langle A_k, X \rangle \leq b_k$$

$X \geq 0$

You should check you can go between forms.

Here's a sketch:

$$\langle A_i, X \rangle \leq b_i$$

$X \geq 0$
\[ x \geq 0 \quad \rightarrow \quad x \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + x \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x & 0 & \cdots & 0 \end{bmatrix} + \cdots + x \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \geq 0 \]

and we've dealt with \( Mx + b \geq 0 \) already.

A problem that is \underline{\text{NOT}} a linear program, but \underline{\text{is}} an SDP.

\[
\begin{align*}
\min \quad & \frac{(c^T x)^2}{d^T x} \\
\text{st.} \quad & Ax + b \geq 0 \\
(\text{where } \quad & d^T x > 0 \text{ for feasible } x ) \\
(\text{fairly obviously not an LP }) \\
(\text{you should check --- how?})
\end{align*}
\]
transform:

\[ \begin{align*}
\text{min} & \quad t \\
\text{st} & \quad Ax + b \geq 0 \\
& \quad \frac{(c^T x)^2}{(d^T x)} \leq t
\end{align*} \]

Trick:

\[ \begin{bmatrix}
  t & c^T x \\
  c^T x & d^T x
\end{bmatrix} \geq 0 - I \]

is:

\[ t + d^T x \geq 0, \quad t (d^T x) - (c^T x)^2 \geq 0 \]

But \( t \geq 0, d^T x \geq 0 \) cause \( d^T x \geq 0 \) for feasible points

So:

\[ I = t \frac{(c^T x)^2}{(d^T x)} \geq 0 \]
Then we have

\[
\begin{align*}
\min & \quad t \\
\text{st} & \quad \begin{bmatrix}
\text{diag}(Ax + b) & 0 & 0 \\
0 & t & (c^T x) \\
0 & (c^T x) & (d^T x)
\end{bmatrix} \succeq 0
\end{align*}
\]

This is equivalent to original and is an SDP.

**Fact:** SDP's can be solved fairly efficiently with interior point methods at moderate scales.
Math trick: the Schur complement.

Consider \[
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix}
= Q \begin{bmatrix}
A & C \\
\end{bmatrix}
\] symmetric

Assume \( A \) is positive definite and solve

\[
Q \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
u \\
v
\end{bmatrix}
\]

\( Ax + By = u \) so \( x = A^{-1}[u - By] \)

\( B^Tx + Cy = v \) so \( B^TA^{-1}[u - By] + Cy = v \)

\[
B^TA^{-1}u + [C - B^TA^{-1}B]y = v
\]

Write \( s \).

So

\[
y = s^{-1} \begin{bmatrix}
v - B^TA^{-1}u
\end{bmatrix}
\]
\( s \) has important applications in positive definiteness.

Consider

\[
\min_u \left[ u^T A u + 2 v^T B u + v^T C v \right]
\]

for fixed \( v \)

\[
u = -A^{-1} B v, \quad \text{so the value is}
\]

\[
v^T B A^{-1} A A^{-1} B v - 2 v^T B A^{-1} B v + v^T C v = v^T S v
\]

This gives \( \text{p.d.}! \)

\[
\begin{aligned}
X \succ 0 & \iff A \succ 0, \; s \succ 0 \\
A \succ 0 & \implies X \succeq 0 \iff s \succeq 0
\end{aligned}
\]

worth remembering!
SDP example:

Quadratically constrained Q.P.

Convex quadratic constraints

\[(Ax+b)(Ax+b) - cx - d \leq 0\]

By Schur complement, this is

\[
\begin{bmatrix}
I & Ax + b \\
(Ax + b)^T & c^T x + d
\end{bmatrix} \succeq 0
\]

(So its a semidefiniteness constraint)

If we have

\[
\text{min } (A_0 x + b)^T (A_0 x + b) - c_0^T x - d_0
\]

\[
\text{st. } (A_i x + b_i)^T (A_i x + b_i) - c_i^T x - d_i \leq 0
\]

Rearrange to get...
\[
\begin{align*}
\text{min} & \quad t \\
\text{s.t.} & \quad \begin{bmatrix} \text{OF block} & 0 & 0 & 0 \\
0 & \text{C}_i \text{ block} & 0 & 0 \\
0 & 0 & \text{C}_i \text{ block} & 0 \\
0 & 0 & 0 & \text{C}_i \text{ block} \end{bmatrix} \geq 0
\end{align*}
\]

Where OF block is
\[
\begin{bmatrix}
I & A_0x + b_0 \\
(A_0x + b_0)^T & C_0x + d_0 + t
\end{bmatrix}
\]

C_i block is
\[
\begin{bmatrix}
I & A_ix + b_i \\
(A_ix + b_i)^T & C_ix + d_i
\end{bmatrix}
\]
Barrier function

\[ \phi(x) = -\log[\det(x)] \]

- if \( x \) has all large eigenvalues
  then \( \phi = -\log[\prod_i \lambda_i] \)
  which grows very slowly w/ \( \lambda \).
- \( \det(x) \to 0 \implies \phi(x) \to +\infty \)

\[ \frac{\partial}{\partial x_{ij}} (\log \det x) = \frac{1}{\det x} \left[ \text{coeff of } x_{ij} \text{ in } \det \right] \]

Sometimes

\[ \frac{\partial}{\partial x} (\log \det x) = x^{-1} \]

(note: this is why large problems might be tough)