A new kind of convex constraint

for F a matrix, write

$$F \succcurlyeq 0 \equiv F \text{ is +ve semidefinite}$$

$$\equiv \underline{x}^T F \underline{x} \geq 0, \text{ for ANY } \underline{x}$$

· this constraint yields a convex set of matrices.

$$F_\alpha \succcurlyeq 0 \quad , \quad F_\beta \succcurlyeq 0$$

$$\Rightarrow \left(t F_\alpha + (1-t) F_\beta \right) \succcurlyeq 0$$

$$(t \in [0,1])$$

[check this — its easy]

The problem:

$$\min_x \quad c^T x$$

$$\text{st} \quad F_0 + \sum_i x_i F_i \succcurlyeq 0$$

symmetric matrices

is a semidefinite program.

- Linear programs are SDP's:

$$\min \quad \underline{e}^T \underline{u}$$
$$\text{st} \quad M\underline{u} = \underline{n}$$
$$P\underline{u} + \underline{q} \geqslant 0$$

Transform: $\underline{u} = \underline{v} + H\underline{x}$ to eliminate equality constraints

to get

$$\min \quad c^T x$$
$$\text{st.} \quad Ax + b \geqslant 0$$

BUT

$$\mathrm{diag}(\underline{v}) \succcurlyeq 0 \iff \underline{v} \geq 0$$

$$Ax + b \geq 0 \iff \mathrm{diag}[Ax + b] \succcurlyeq 0$$

and $A = [a_1, \; \cdots \; a_r]$ ← cols.

and

$$\mathrm{diag}[Ax + b] \succcurlyeq 0$$
$$\parallel$$
$$\mathrm{diag}(b) + \sum_i x_i \,\mathrm{diag}(a_i) \succcurlyeq 0$$

so $\min \; c^T x$

$\quad$ st $\;\mathrm{diag}(b) + \sum_i x_i \,\mathrm{diag}(a_i) \succcurlyeq 0$

which is an SDP

SDP's take a variety of forms

for matrices A, X write

$$\langle A, X \rangle = \mathrm{Trace}\left[A^T B\right] = \sum_{ij} A_{ij} B_{ij}$$

check this equivalence - its useful, and you'll see it often.

SDP's can be written as

$$\min_X \ \langle C, X \rangle$$

$$\text{s.t.} \quad \langle A_K, X \rangle \leq b_K$$

$$X \succcurlyeq 0$$

You should check you can go between forms.
Heres a sketch:

this is x, straightened into a vector

$$\begin{matrix} \langle A_i, X \rangle \leq b_i & \longrightarrow & Mx + n \geq 0 \\ X \succcurlyeq 0 & & X \succcurlyeq 0 \end{matrix}$$

$$X \succeq 0 \longrightarrow x_{11}\begin{bmatrix} 1 & 0 & \cdot \\ 0 & 0 & \\ \cdot & & \cdot \end{bmatrix} + x_{12}\begin{bmatrix} 0 & 1 & \cdot \\ 0 & 0 & \\ \cdot & & \cdot \end{bmatrix}$$

$$+ \cdots \succeq 0$$

and we've dealt with $Mx + b \geq 0$ already.

- A problem that is NOT a linear program, but is an SDP.

$$\min \quad \frac{(c^T x)^2}{d^T x}$$

$$\text{s.t.} \quad Ax + b \geq 0$$

(where $d^T x > 0$ for feasible x)

(fairly obviously not an LP)

(You should check — how?)

transform:

$$\min \quad t$$

$$\text{st} \quad Ax + b \geq 0$$

$$\frac{(c^T x)^2}{(d^T x)} \leq t$$

Trick:

$$\begin{bmatrix} t & c^T x \\ c^T x & d^T x \end{bmatrix} \succeq 0 \qquad \text{— I}$$

is: $\quad t + d^T x \geq 0 , \qquad t(d^T x) - (c^T x)^2 \geq 0$

But $t \geq 0, d^T x > 0$ cause $d^T x > 0$ for feasible points

So: $\quad \text{I} \equiv \quad t - \dfrac{(c^T x)^2}{(d^T x)} \geq 0$

Then we have

$$\min \quad t$$

$$\text{st} \quad \begin{bmatrix} \operatorname{diag}(Ax+b) & 0 & 0 \\ 0 & t & (c^T x) \\ 0 & (c^T x) & (d^T x) \end{bmatrix} \succeq 0$$

This is equivalent to original and is an SDP.

<u>Fact</u>: SDP's can be solved fairly efficiently with interior point methods, at <u>moderate scales</u>

Math trick: the Schur complement.

consider $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = Q$ [A, C symmetric

assume A is positive definite and solve

$$Q\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$Ax + By = u$ so $x = A^{-1}[u - By]$

$B^T x + Cy = v$ so $B^T A^{-1}[u - By] + Cy = v$

$$B^T A^{-1} u + \underbrace{[C - B^T A^{-1} B]}_{\text{write } S} y = v$$

so $$y = S^{-1}\left[v - B^T A^{-1} u\right]$$

S has important applications in positive definiteness

consider

$$\min_u \left[u^T A u + 2 v^T B^T u + v^T C v \right]$$

<u>for fixed v</u>

$u = -A^{-1} B v$, so the <u>value</u> is

$$v^T B^T A^{-1} A A^{-1} B v - 2 v^T B^T A^{-1} B v + v^T C v = v^T S v$$

this gives (P.D.!)

$$\boxed{\begin{array}{l} X \succ 0 \iff A \succ 0,\ S \succ 0 \\ A \succ 0 \implies \boxed{X \succeq 0 \iff S \succeq 0} \end{array}}$$

worth remembering!

SDP example:

Quadratically constrained Q.P.

convex quadratic constraints

$$(Ax+b)^T(Ax+b) - c^Tx - d \leq 0$$

By Schur complement, this is

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^Tx+d \end{bmatrix} \succcurlyeq 0$$

(so its a semidefiniteness constraint)

If we have

$$\min \; (A_0x+b)^T(A_0x+b) - c_0^Tx - d_0$$

$$\text{s.t.} \; (A_ix+b_i)^T(A_ix+b_i) - c_i^Tx - d_i \leq 0$$

rearrange to get ..

$$\min \; t$$

s.t.

$$\left[\begin{array}{c|c|c} \text{OF block} & 0 & 0 \\ \hline 0 & C_1 \text{ block} & 0 \\ \hline 0 & 0 & C_i \text{ block} \end{array}\right] \succcurlyeq 0$$

<u>Where</u> OF block is $\begin{bmatrix} I & A_0x + b_0 \\ (A_0x+b_0)^T & c_0^Tx + d_0 + t \end{bmatrix}$

C_i block is $\begin{bmatrix} I & A_ix + b_i \\ (A_ix+b_i)^T & c_i^Tx + d_i \end{bmatrix}$

What about SDP in interior point methods

- Barrier function

$$\phi(X) = -\log\left[\det(X)\right]$$

  - if X has all large eigenvalues then $\phi = -\log\left[\prod_i \lambda_i\right]$ which grows very slowly w/ λ.
  - $\det(X) \xrightarrow{+} 0 \Rightarrow \phi(X) \to +\infty$

gradients

$$\frac{\partial}{\partial x_{ij}}\left(\log \det X\right) = \frac{1}{\det X}\left[\text{coeff of } x_{ij} \text{ in } \det\right]$$

sometimes

$$\frac{\partial}{\partial X}\left(\log \det X\right) = X^{-1}$$

(notice: this is why large problems might be tough)