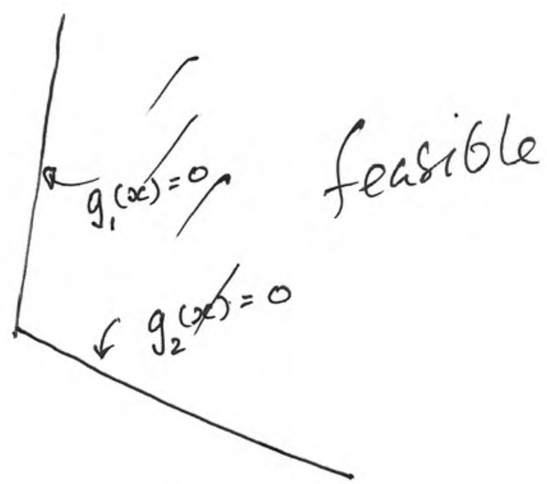
Lagrangians and inequalities



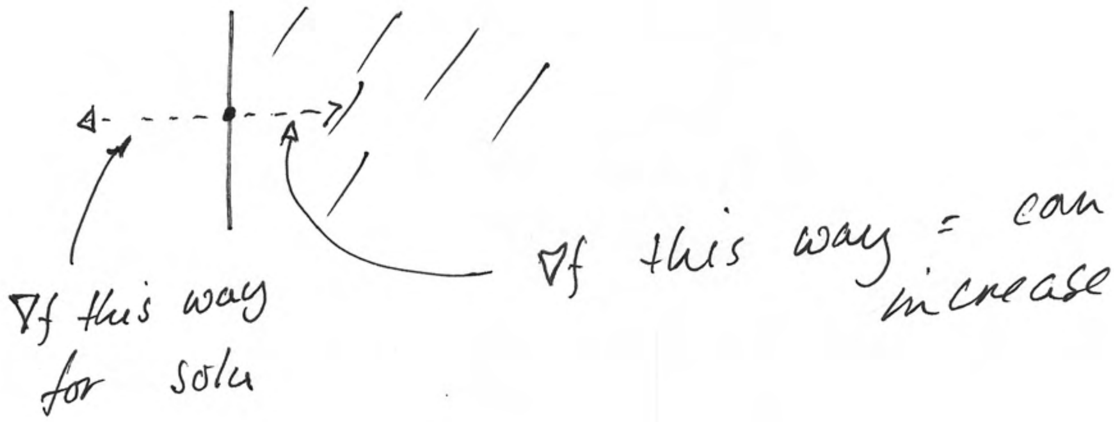
consider $\max f(x)$

st $g(x) \geq 0$

<u>cases</u>: interior — constraints don't operate

boundary — one can move <u>along</u>, into interior, but not <u>across</u>

max f st $g \geq 0$



i.e. at soln

$$\nabla f = \lambda \nabla g$$

$$\lambda \geq 0$$

and $g \geq 0$

and if $g = 0$, $\lambda > 0$

$g > 0$, $\lambda = 0$

and this applies for multiple constraints

Lagrangians for mixed probs

$$\max f \quad \text{st} \quad \begin{aligned} g &\geq 0 \\ h &= 0 \end{aligned}$$

$$\mathcal{L}: \quad f(x) \mp \lambda^T g \mp \mu^T h$$

Conditions:

$$\boxed{\begin{aligned} \nabla_x \mathcal{L} &= 0 \\ g &\geq 0 \\ h &= 0 \\ \lambda_i \cdot g_i &= 0 \\ \lambda &\geq 0 \end{aligned}}$$

Karush - Kuhn-Tucker Conditions or KKT.

complementarity Condition - source of <u>much</u> mischief

VICIOUSLY non-linear

$$\equiv \left\{ \begin{array}{ll} g = 0, & \lambda > 0 \\ \text{OR} & \\ g > 0, & \lambda = 0 \end{array} \right\}$$

We can still construct duals

$$Q(\lambda, \nu) = \sup_x \mathcal{L}(x, \lambda, \nu)$$

(notice: max gives sup in dual
min gives inf
saddle point arg still works.

$$Q(\lambda, \nu) \geq f(x^*)$$ solution

So gap reasoning works, too.

Constructing duals:

Linear program:

$$\begin{aligned} \max \quad & c^T x \\ \text{st} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

~~QL~~

$$\mathcal{L}(x, \lambda, \cancel{\mu}) = c^T x - \lambda^T \cancel{x} - \mu^T (Ax - b)$$

$$Q(\lambda, \nu) = \cancel{\sup} \inf_x \mathcal{L}(x, \lambda, \nu)$$

this is infinite <u>unless</u> the coeff of x in $\mathcal{L}$ $= 0$

$$\therefore \boxed{c \; \cancel{c} - A^T \mu - \lambda = 0}$$

domain where dual exists

In this domain, $\mathcal{L}(x, \lambda, \nu) = \mu^T b$

and $\lambda \geq 0$ (from original.

So dual is

$$\begin{aligned} \min \quad & \mu^T b \\ \text{s.t.} \quad & c - A^T \mu - \lambda = 0 \\ & \lambda \geq 0 \end{aligned}$$

Notice different forms have different forms of Dual

e.g.

$$\begin{aligned} \max \quad & c^T x \\ \text{st} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$\Big\downarrow \text{Dual}$$

$$\begin{aligned} \min \quad & b^T \mu \\ \text{st} \quad & A^T \mu \geq c \\ & \mu \geq 0 \end{aligned}$$

KKT link primal and dual

e.g. L.P. case.

(x, λ, μ) is
(a) Primal feasible
(b) Dual feasible
(c) Complementarity

$$\Updownarrow$$

(x, λ, μ) satisfy KKT

$$\Updownarrow$$

(x, λ, μ) are soln.

This is just bookkeeping, but it's powerful.

This opens a new world of algorithmic possibilities

<u>Simplex</u> – work in primal.

<u>New</u> · <u>A</u>· – work with primal feasible x, dual infeasible λ, μ complementary.

<u>B</u> : – work w/ primal ~~in~~feasible x dual feasible λ, μ complementary.

<u>C</u> : – work w/ primal feasible x dual feasible λ, μ Not complementary.

In each, adjust working point to ~~[illegible]~~ meet missing properties better.