Conjugate direction methods:

- A set of vectors, $p_0, \ldots, p_n$, is conjugate for $A$ positive definite if $p_i' A p_j = 0$ if $i \neq j$.

- Assume we wish to $\min \frac{x' A x - b' x}{2}$

- Useful because:
  a) Solution to $A x = b$ for $A$ p.d.
  b) $\min_x \| x - b \|^2$ is like this

- Now write

$$x = \alpha_0 p_0 + \alpha_1 p_1 + \ldots$$
Conjugate direction in incremental form

Start with \( x_0, p_0 \)

\[
x_1 = x_0 + \alpha_0 p_0
\]

now write \( \alpha_0 \)

to get

\[
\frac{(Ax_0 - 6)' p_0}{p_0' A p_0} = \alpha_0
\]

write

\[
r_K = (Ax_K - 6)
\]

and get

\[
x_{K+1} = x_K + \alpha_K p_K
\]

\[
\alpha_K = \frac{r_K' p_K}{p_K' A p_K}
\]

\[
r_{K+1} = r_K + \alpha_K A p_K
\]
This gives

\[
\frac{1}{2} \left[ (x_k + \alpha_k p_k)' A (x_k + \alpha_k p_k) \right] \\
- b_k' (x_k + \alpha_k p_k)
\]

and is at:

\[
-(A x_k - b)' p_k \\
\frac{p_k' A p_k}{p_k' A p_k}
\]

write

\[
r_k = A x_k - b
\]

so \[\alpha_k = -r_k' p_k \\
\frac{p_k' A p_k}{p_k' A p_k} \]
gate gradient (simple form)

Start: \( x_0, r_0 = A x_0 - b, p_0 = -r_0 \)

Step:

\[
\begin{align*}
x_{k+1} &= x_k + \alpha_k p_k \\
\alpha_k &= -r_k^T p_k \\
&\quad / p_k^T A p_k \\
p_{k+1} &= -r_{k+1} + \beta_{k+1} p_k \\
\beta_{k+1} &= r_{k+1}^T p_k \\
&\quad / p_k^T A p_k \\
r_{k+1} &= r_k + \alpha_k A p_k
\end{align*}
\]
Properties of conj. direction

\[ r_k' p_i = 0, \quad \forall i < k \]

(Show this by induction)

\[ r_k' r_i = 0 \quad \forall i < k \]

(thus 5.3 at end)
Conjugate gradient, cleaner form:

By properties, we have

\[ \alpha_{k+1} = \frac{\bar{r}_k' \bar{r}_k}{p_k' A p_k} \]

Now \( \alpha_k A p_k = \bar{r}_{k+1} - \bar{r}_k \)

So

\[ \beta_{k+1} = \frac{\bar{r}_{k+1}' (\bar{r}_{k+1} - \bar{r}_k)}{\alpha_k} \cdot \frac{1}{p_k' A p_k} \]

\[ = \frac{\bar{r}_{k+1}' (\bar{r}_{k+1} - \bar{r}_k)}{\bar{r}_k' \bar{r}_k} \]

\[ = \frac{\bar{r}_{k+1}' \bar{r}_{k+1}}{\bar{r}_k' \bar{r}_k} \quad (\text{By properties}) \]