Max cut, SDP and approximation

Max cut: given $G = (V, E)$ find a Maximum cut for weights w_{ij}

i.e. $S, \hat{S}$ st $S \cap \hat{S} = \emptyset$

$S \cup \hat{S} = V$

$$\sum_{i \in S \, j \in \hat{S}} w_{ij}$$

- NP-complete
- NP-complete if $w_{ij} = 1$

! • Polynomial if G is planar and in some other special cases.

- 0-1 quadratic form

write $x_i = \begin{cases} 0 & i \in S \\ 1 & i \in \hat{S} \end{cases}$

~~then~~ max cut is

$$\max \sum_{ij} (1-x_i)\, w_{ij}\, x_j$$

$$x_i = \{0, 1\}$$

i.e. $\max \quad \dfrac{x^T A x}{2} + b^T x$

$$\text{st} \quad x_i \in \{0, 1\}$$

and A's elements (are / could be) negative

NOTE: flip an unbiased coin at each vert gives $E(\text{value}) = \frac{1}{2} \cdot$ Optimal.

Max cut:

$$\frac{1}{2} \sum_{i<j} w_{ij} (1 - y_i y_j)$$

$$y_i \in \{-1, 1\}$$

RELAX:

R $\longrightarrow$ replace $y_i y_j$ with

$v_i \cdot v_j$

subject to $v_i \cdot v_i = 1$, for all i

Approximation algorithm

1. solve R to get v_i
2. consider r, uniform at random on sphere
3. $S = \{ i \mid v_i \cdot r \geq 0 \}$

- This approximation comes with a STARTLING guarantee.

Write $E(W)$ = Expected value of soln to approx problem

Z^*_{MC} = value of exact soln.

Z^*_R = value of ~~approx~~ R.

Then clearly $Z^*_R \geq Z^*_{MC}$

But we can show

$$E[W] = \sum_{i<j} w_{ij} \left(\frac{\arccos v_i \cdot v_j}{\pi} \right)$$

and: $E[W] \geq \alpha Z^*_R = \alpha \cdot \frac{1}{2} \sum_{i<j} w_{ij} (1 - v_i \cdot v_j)$

$$\alpha \simeq 0.87856$$

SO

$$E[W] \geq \alpha Z^*_{MC}.$$

Semi Definite programming,

or how to solve R.

SDP:

$$\min \quad c^T x$$

s.t. $\quad \underbrace{F(x) \succeq 0}$

↳ is positive Semi Definite

and $\quad F(x) = F_0 + \sum_i x_i F_i$

: symmetric matrices

- this problem is convex

(obvious that objective is convex)

Domain is convex because

$$F_\alpha \succeq 0, \quad F_\beta \succeq 0$$

$$\Rightarrow \quad t F_\alpha + (1-t) F_\beta \succeq 0$$

$$t \in [0,1]$$

Cases :

· **Linear program:**

$$\min \quad c^T x$$

$$\text{st} \quad Ax + b \geq 0$$

↑ component wise inequality

Now $\quad \operatorname{diag}(v) \succeq 0 \iff v \geq 0$

$\therefore \quad Ax + b \geq 0 \iff \operatorname{diag}[Ax + b] \succeq 0$

$$A = [a_1 \; \cdots \; a_n]$$

↑ cols

$$\operatorname{diag}[Ax + b] = \operatorname{diag}(b) + \sum_i x_i \operatorname{diag}(a_i)$$

So $\quad \min \quad c^T x$

$$\text{st} \quad \operatorname{diag}(b) + \sum_i x_i \operatorname{diag}(a_i) \succeq 0$$

- A problem that is an SDP but not an LP

(I)
$$\min \frac{(c^T x)^2}{d^T x}$$

$$\text{st} \quad Ax + b \geq 0$$

(where $d^T x > 0$ for ~~the~~ x s.t. $Ax + b \geq 0$)

- this is convex, Not LP.

(II)
$$\min \quad t$$

$$\text{st} \quad Ax + b \geq 0$$

$$\frac{(c^T x)^2}{(d^T x)} \leq t$$

$I \equiv II$ Trick:

$$\begin{bmatrix} t & (c^T x) \\ (c^T x) & (d^T x) \end{bmatrix} \succeq 0$$

is: $\quad t(d^T x) - (c^T x)^2 \geq 0$

but $\quad d^T x > 0$ by assumption, so $t - \frac{(c^T x)^2}{d^T x} \geq 0$

So $\quad \min \quad t$

III $\quad$ st $\begin{bmatrix} \text{diag}(Ax+b) & 0 & 0 \\ 0 & t & (c^T x) \\ 0 & (c^T x) & (d^T x) \end{bmatrix} \succcurlyeq 0$

is equivalent to I ; but III is an SDP.

Fact: SDP's can be solved fairly efficiently with interior point methods (estimates later).

~~Further examples:~~

~~Quadratically constrained Quadratic program~~

Schur complements

consider

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$ A, C, Symmetr

and assume A positive definite

now

$$X \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$Ax + By = u$$

$$B^T x + Cy = v$$

so: $$x = A^{-1} u - A^{-1} B y$$

and $$B^T A^{-1} u - B^T A^{-1} B y + Cy = v$$

write $$S = C - B^T A^{-1} B$$

$$B^T A^{-1} u + S y = v$$

$$y = S^{-1}(v - B^T A^{-1} u)$$

S is very interesting, w/ applications to positive definiteness.

Consider

$$\min_u \left[u^T A u + 2 v^T B^T u + v^T C v \right]$$

(for fixed v)

then $u = -A^{-1} B v$ so the value is

$$v^T B^T A^{-1} A A^{-1} B v - 2 v^T B^T A^{-1} B v + v^T C v$$

$$= v^T S v$$

this gives

$$\boxed{\begin{array}{l} X \succeq 0 \iff A \succ 0,\ S \succ 0 \\ A \succ 0 \Rightarrow \left[X \succeq 0 \iff S \succeq 0 \right] \end{array}}$$

worth remembering!

SDP example: Quadratically constrained Q.P.

assume convex quadratic constraints

$$(Ax+b)^T(Ax+b) - c^Tx - d \preceq 0$$

By Schur complement equiv to

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^Tx+d \end{bmatrix} \succeq 0$$

this has right form!

so if we have

minimize $\quad (A_0x+b_0)^T(A_0x+b_0) - c_0^Tx - d_0$

s.t.

$$(A_ix+b_i)^T(A_ix+b_i) - c_i^Tx - d_i \preceq 0$$

we can rearrange to get

min $\quad t$

subject to

$$\begin{bmatrix} \text{OF block} & 0 & 0 & 0 & \cdots \\ 0 & C_1 \text{ block} & 0 & \cdots & \\ 0 & 0 & C_2 \text{ block} & & \\ \vdots & & & \ddots & \end{bmatrix} \succeq 0$$

<u>where</u>

$$C_i \text{ block} = \begin{bmatrix} I & A_i x + b_i \\ (A_i x + b_i)^T & c_i x + d_i \end{bmatrix}$$

$$\cancel{C_0}\ \text{OF block} = \begin{bmatrix} I & A_0 x + b_0 \\ (A_0 x + b_0)^T & c_0 x + d_0 + t \end{bmatrix}$$

we can now write R as an SDP

$\underline{R}$ $\min \sum_{i<j} w_{ij} (1 - v_i \cdot v_j)$

s.t. $v_i \cdot v_i = 1$

- Consider

$$V = [v_1 \quad \cdot \quad \cdot \quad v_n]$$

$\mathrm{diag}(V^T V) = 1$

$V^T V \succeq 0$

~~then~~ AND for $Y \succeq 0$, $\mathrm{diag}(Y) = 1$

we can find V by cholesky fact.

$\underline{R'}$: $\min \sum_{i<j} w_{ij} (1 - y_{ij})$

s.t. $y_{ii} = 1$

$Y \succeq 0$

(we can subst $y_{ii} = 1$ into prob to get our SDP form)

Actually, we can attack $\{-1,1\}$ Quadratic forms in general.

· consider

$$x^T A_0 x + b_0^T x$$

$$\text{st} \quad x_i^2 = 1 \qquad x^T A_i x + 2b_i^T x + c_i \le 0$$

{and we can get from/to $x_i \in \{-1, 1\}$ to/from $u_i \in \{0,1\}$ by a simple linear subst}

· could write as

$$\min \quad \mathrm{Tr}\, X A_0 + b_0^T x$$

$$\text{st} \quad x_i^2 = 1$$

$$\mathrm{Tr}\, X A_i + b_i x + c_i \le 0$$

$$x x^T = X$$

Relaxation: (which seems to be successful)

Relax

$$xx^T = X$$

to

$$X - xx^T \succcurlyeq 0$$

which is:

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succcurlyeq 0$$

SDP relaxations are not widely used in vision (too many vars in practice, tend to be slow)