More on bipartite matching:

- Assume we are dealing with n verts on each side, a complete bipartite graph
- Then we can think about matchings as **permutations** e_i goes to $\pi(e_i)$, etc.
- Recall you can write a permutation as a 0-1 matrix M, with some special properties

$$M_{ij} \in \{0, 1\}$$

$$\sum_i M_{ij} = 1$$

$$\sum_j M_{ij} = 1$$

Now recall the Birkhoff - Von Neumann thm:

- Consider the set D of doubly-stochastic matrices (square matrices A such that $\sum_i A_{ij} = 1$, $\sum_j A_{ij} = 1$)

- D is convex
- D is the convex hull of the permutation matrices
- The vertices of D are exactly the permutation matrices (no other verts).

We can see maximum weighted bipartite matching as

$$\max \sum_{ij} w_{ij} M_{ij}$$

$$\text{s.t.} \qquad M_{ij} \in \{0,1\}$$

$$\sum_i M_{ij} = 1 \qquad \sum_j M_{ij} = 1$$

__BUT__ if we relax this to

$$\max \sum_{ij} w_{ij} M_{ij}$$

$$M \in D$$

The solution to the relaxed problem is a soln to the orginal problem (Because the verts of D are integer)

Considerably more is possible:

- Consider a bipartite problem where there are different numbers of points on each side (ie can't do B-vN then).

- We can represent the mathing w/ an ~~u~~ $u \times v$ matrix M st

$$\sum_{i \in u} M_{ij} \leq 1$$

$$\sum_{j \in v} M_{ij} \leq 1$$

$$M_{ij} \in \{0, 1\}.$$

- Now turn M into a <u>vector</u> m by rearranging, etc.

- Write constraints as

$$Am \leq 1$$

- Think about structure of ~~m~~ A
  - each <u>col</u> of A corresponds to an entry in M.
  - each entry in M appears in <u>exactly</u> 2 inequalities

$\Rightarrow$ every col of A contains precisely 2 ones.

Thm A is TUM.

Proof: (Induction)

Take a square submatrix T

3 cases

- Some col is all zeros
- at least one col contains only one 1
  → pass to smaller submatrix
- every col contains 2 ones.

- In this case, each row of A corresponds to either a row constraint on M or a col constraint on M.

- But there must be the same # of each

~~So there is~~

· So T can be permuted to

$$
\begin{array}{l|c|l}
\text{rows csp. row const.} & \left\{ \qquad\qquad\qquad \right. & \updownarrow k \\
 & \left\{ \qquad\qquad\qquad \right\} & \updownarrow k \ \text{rows csp. col constraints}
\end{array}
$$

~~So~~ But one of the ones in each col appears in the first block, and the other in the second block.

So
$$
T \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{array} \right] \begin{array}{l} \\ \updownarrow k \\ \\ \\ \updownarrow k \\ \\ \end{array} = 0
$$

So $\det(T) = 0$

(By the way, this doesn't mean all minors $= 0$)

So we can write weighted bipartite matching as

$$\max \quad w^T m$$

$$\text{st} \quad Am \leq 1$$

$$m \geq 0$$

$$m \in \{0,1\}$$

<u>BUT</u> A is TUM

So we can <u>relax</u> to

$$\max \quad w^T m$$

$$\text{st.} \quad m \geq 0$$

$$Am \leq 1$$

$$\cancel{m \in \{0,1\}}$$

and the soln to this is <u>integer</u>