Miterm Exam Answer Sheet.

Q1. Two fools decide to hold a gunfight, but have only one gun (a six cylinder resolvers) and one bullet. They agree to fight according to the following scheme: A will pick up the gun, spin the cylinder, and fire at B (with a probability of 1/6 that the hammer will fall on a live round). If the gun fires, A wins. If the gun does not fire, B will pick up the gun, spin the cylinder, and fire at A (with a probability 1/6 that the hammer will fall on a live round). If the gun fires this time, B wins. Otherwise, it's a draw.

[Hint: A starts to fire the gun, so A wins with probability $\frac{1}{6}$ that the gun fires successfully; otherwise B takes the turn, under another probability $\frac{1}{6}$ B wins; otherwise it's a draw.]

(a). What is the probability that A wins?

$$\frac{1}{6}$$

(b). What's the probability that B wins?

$$(1-\frac{1}{6})\cdot\frac{1}{6}.$$

(c). They find the idea of a draw frustrating. They decide to resolve this by simply repeating the procedure if there is a draw until the gun fires. What's the number of trigger pulls?

$$P(x=n)=(1-\frac{1}{6})^{n-1}\frac{1}{6}$$

It is a geometric distribution with $p=\frac{1}{6}$, so

$$E_p(x)=1/p=6$$

Q2. A dataset consists of 6 IID samples from a probability distribution $P(x)$. The dataset is $\{0, 1, 1, 0, 0, 1\}$

(a). Estimate $E[X]$

$$E[X] = \frac{0+1+1+0+0+1}{\#samples} = 3/6 = 0.5$$

(b). Estimate $E[X^2]$

$$E[X] = \frac{0^2+1^2+1^2+0^2+0^2+1^2}{\#samples} = 3/6 = 0.5$$

(c). Estimate $Var[X]$

$$Var[X] = E[X^2] - E^2[X] = 0.5 - 0.5^2 = 0.25$$

(d). Why can you make estimates without knowing $P(X)$ (short phrase)?

The dataset is *IID samples*.

Q3. Teh patriot missile is an anti-missile missile that was deployed in the first gulf war. At that time the Pentagon claimed that the patriot missile had a probability of destroying a target of 0.8. They released video tapes of 14 missles firings; 13 showed the missile failing, and one showed the missile destropying its target. Assuming the pentagon's estimate of the probability of destropying a target is correct.

(a). Write an expression for the probability of 1 success in 14 independant firings.

$$\binom{14}{1} * 0.8 * 0.2^{13}$$

(b). Is this number smaller than 10^{-6}?
Yes.

$$\binom{14}{1} * 0.8 * 0.2^{13} = \frac{14 * 0.8 * 2^{13}}{10^{13}} = \frac{14 * 0.8 * 8 * 1024}{10^{13}} < \frac{10^7}{10^{13}} = 1e - 6. \quad (1)$$

Q4. A machine has k working parts. Each part will fail with probability 1/3 in year 1 and with probability 2/3 in year 2. Each failure is independant.

[Hint: I think David expained in class that "Each part will fail with probability 1/3 in year 1 and with probability 2/3 in year 2" means if a part does not fail in year 1, then it must fail in year 2 (because 1/3 + 2/3 = 1, this means the probability that a part fails in year 2 is not independtant of its status in year 1). There is a different way to read this if we assume a part fails in year 2 with probability independant of its status in year 1.]

(a). Assume that if any part fails, the machine becomes inoperable. Write an expression for the expected number of years before it becomes inoperable.

$$P(X=1) = 1-(1-\frac{1}{3})^k = 1-(\frac{2}{3})^k, \quad P(X=2) = 1-P(X=1) = (\frac{2}{3})^k$$

$$E[X] = 1P(X=1) + 2P(X=2) = 1 + (\frac{2}{3})^k$$

(b). Now assume that a clever designer rearranges the machine so that it becomes inoperable only if all parts fail. Write an expression for the expected number of years before it becomes inoperable?

$$P(X=1) = (\frac{1}{3})^k, \quad P(X=2) = 1-P(X=1) = 1-(\frac{1}{3})^k$$

$$E'[X] = 1P(X=1) + 2P(X=2) = 2 - (\frac{1}{3})^k$$

(c). Which machine is more reliable?

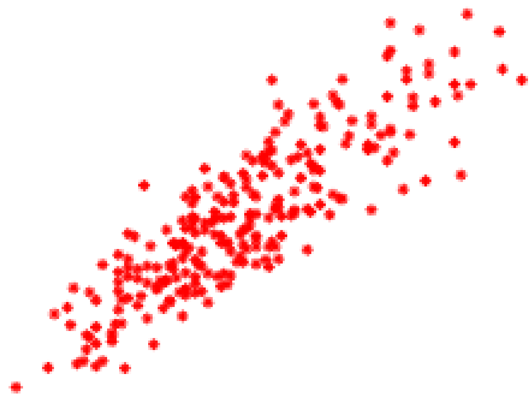
The second machine when k is greater than 1 (intuitively); or mathematically:

$$E'[X] - E[X] = 1 - ((\frac{1}{3})^k + (\frac{2}{3})^k) \geq 1 - (\frac{1}{3} + \frac{2}{3})^k = 0$$

Q5. You select three people at random, and ask the month of their birth. Each month has the same probability of being the month of the birth for a randomly selected person. You could be told 1 (i.e. all born in the same month), 2 (i.e. two born in one month, one born in a different), or 3 distinct months. Write an expression for the expected number of months?

$$P(X=1) = \binom{12}{1} * \frac{1}{12} * \frac{1}{12} * \frac{1}{12} = \frac{1}{144};$$
$$P(X=2) = \binom{3}{2} * (\binom{12}{1}\frac{1}{12} * \frac{1}{12}) * \frac{11}{12} = \frac{33}{144};$$
$$P(X=3) = \frac{11}{12} * \frac{10}{11} = \frac{110}{144}$$

$$E[X] = 1P(X=1) + 2P(X=2) + 3P(X=3) = \frac{397}{144}$$



Q6. The figure shows a scatter plot of a dataset in normalized coordinates.

(a). What is the sign of r, the correlation coefficient?
Positive

(b). The magnitude of r is 0.8, the x-coordinate of a data item in normalized coordinates in 1, what's the best prediction of its y-coordinate

$$\hat{y} = rx = 0.8 * 1 = 0.8$$

(b). The magnitude of r is 0.8, the y-coordinate of a data item in normalized coordinates in -0.2, what's the best prediction of its x-coordinate

$$\hat{x} = ry = 0.8 * (-0.2) = -0.16$$