

# Recognizing Algebraic Surfaces from their Outlines

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Received ??, Revised ??.

## Abstract.

The outline in a single picture of a generic algebraic surface of degree three or greater completely determines the projective geometry of the surface. The result holds for a generic perspective view of a generic algebraic surface, where the camera calibration parameters and the focal point are unknown. Known camera calibration appears not to reduce the projective ambiguity. The result is constructive.

**Keywords:** Recognition, Computer Vision, Algebraic Surfaces, Invariant Theory, Outlines.

## 1. Introduction

Outlines, the points in an image where a surface turns away from the camera, are a potentially important source of information about the objects in a scene. Typically, image edges appear at most outline points, and image edges can be computed reasonably reliably. This potential has not been realised in the case of curved surfaces, because the complicated relationship between the outline of a curved surface and the surface makes outlines hard to interpret. This paper shows that, although the relationship between surface and outline is complicated, for a large class of surfaces the outline is sufficiently highly structured to determine the surface's projective geometry from a single view.

### 1.1. Recognising curved surfaces

There have been many approaches to recovering shape information for curved surfaces from images, including attempts to extend line labelling to curved shapes (e.g. Freeman and Shapira, 1978; Malik, 1987), the development of constraint-based systems (e.g. Brooks, 1983), the study of how the topology of a surface's outline changes as it is viewed from different points, formalised into a structure known as an *aspect graph* (for example, Koenderink, 1987; Koenderink, 1992;

Plantinga and Dyer, 1987; Ponce and Kriegman, 1992; Rieger, 1992) and studies of the relationship between the differential geometry of the outline and that of the surface, for single images (e.g. Koenderink and Van Doorn, 1979; Marr, 1982; Koenderink, 1984) as well as for motion sequences (e.g. Giblin and Weiss, 1986; Cipolla and Zisserman, 1992).

Each approach has characteristic disadvantages: extensions to line labelling and aspect graphs can be extremely complicated for even simple curved surfaces (examples in Petitjean *et al.*, 1992; Ponce and Kriegman, 1992; Rieger, 1992); constraint-based systems must search a model-base, and studies of differential properties seldom yield sufficient information to identify a surface. Recently, there have been attempts to represent the system of outlines of a curved surface as a linear combination of outlines (see, for example, Ullman and Basri, 1991). This approach is represented as providing an approximation sufficiently accurate for some purposes, although it cannot capture all the complexities of the outline. There are two main difficulties: it is hard to specify correspondences between outline points, so that the linear combination is ill-defined; and, because the scheme is based on a local approximation, it cannot capture the global interactions between a surface and a focal point that produce the outline. However, the approximation can yield plausible

outlines when views are taken from similar viewing positions.

Recovering surface geometry from a single outline is intractable if the surface is constrained only to be smooth or piecewise smooth, because significant changes can be made to the surface geometry without affecting the outline from a given viewpoint. As a result, an important part of the problem involves constructing as large a class of surfaces as possible that can either be directly recognised or usefully constrained, from their outline alone. In this context, studies have focused on rotationally symmetric surfaces (for example, Dhome *et al.*, 1990; Forsyth *et al.*, 1992) and straight homogenous generalised cylinders (for example, the work of Binford *et al.*, 1989; Ponce, 1989; Ulupinar and Nevatia, 1990 and 1991; Zerroug and Nevatia, *in press*)

Recent work of Ponce and Kriegman (in Ponce and Kriegman, 1989, 1990a, 1990b; Ponce *et al.*, 1991; Ponce and Kriegman, 1992) focussed attention on *algebraic surfaces*. Algebraic surfaces, which consist of all the points in space where a single polynomial vanishes, have numerous advantages as objects of study:

- Many man-made surfaces are made up of “patches” of algebraic surface, as most popular CAD/CAM surfaces are algebraic.
- The geometry of an algebraic surface is determined by a relatively small number of parameters (the coefficients of the polynomial that gives the surface). At the same time, the surfaces have a rich and useful geometry.
- Algebraic surfaces have important “rigidity” properties. For example, one cannot add a local bump to an algebraic surface and obtain another algebraic surface; the whole surface must be deformed instead.

Ponce and Kriegman showed that elimination theory can be used to predict the outline of an algebraic surface viewed from an arbitrary viewing position. For a given surface a viewing position is then chosen using an iterative technique, to give a curve most like the curve observed. The object is then recognized by searching a database, and selecting the member giving the best fit to the observed outline. This work shows that outline curves strongly constrain the viewed surface, but has the disadvantage that it cannot recover sur-

face parameters without solving an optimization problem, so that for a big model base, each model may have to be tested in turn against the image outline. Furthermore, camera parameters must be known to predict the outline correctly.

## 1.2. Indexing for recognition

A number of recent papers have shown how *indexing* can be used to avoid searching a model base (e.g. Lamdan *et al.*, 1988; Forsyth *et al.*, 1991; Rothwell *et al.*, 1991; Wayner, 1991; Stein and Medioni, 1992). Objects are indexed by computing descriptions that are unaffected by the position and intrinsic parameters of the camera, and that differ from object to object. These descriptions, often known as *indexing functions*, have the same value for any view of a given object, and so can be used to index into a model base without search.

In a typical system that works for plane objects, projective invariants<sup>1</sup> are computed for a range of geometric primitives in the image. If the values of these invariants match the values of the invariants for a known model, we have good evidence that the image features are within a camera transformation of the model features. As a result, these invariants index into a model base directly. Object models consist of a set of invariant values and are therefore relatively sparse, meaning that hypothesis verification is required to confirm a model match. However, no searching of the model base is required because the hypothesised object’s identity is determined by the invariant descriptors measured. Systems of this sort have been demonstrated for plane objects in a number of papers (Forsyth *et al.*, 1991), (Lamdan *et al.*, 1988), (Rothwell *et al.*, 1992), (Taubin and Cooper, 1992), (Weiss, 1988), (Zisserman *et al.*, 1992). These systems are attractive because, in the ideal case, an object description is computed from the image and identifies the object, without requiring that a model base be searched. As a result, systems with relatively large model bases can be constructed<sup>2</sup>.

In the case of plane objects, indexing functions are easy to compute, because a view of a plane curve from an arbitrary focal point is within a projective transformation of the origi-

nal curve. Constructing indexing functions for three dimensional objects is challenging, because a change in viewing position can lead to a profound change in the geometry of the outline. Furthermore, any indexing function should be both *invariant* and *computable from outline information alone*. Indexing functions with these properties have been demonstrated for polyhedra (Rothwell *et al.*, 1993), and for rotationally symmetric surfaces (Forsyth *et al.*, 1992).

This paper shows that such indexing functions can be computed for algebraic surfaces viewed in perspective using an uncalibrated camera, by establishing:

**THEOREM 1** *The equation of its outline in a perspective image completely determines the projective geometry of an algebraic surface of degree 2 or greater, for a generic view of a generic algebraic surface.*

Here generic means “almost every”; precisely, the generic algebraic surfaces are all algebraic surfaces except those whose coefficients satisfy a non-trivial system of algebraic relations, to be determined later. At this point, we assume that the surface is smooth and irreducible. The result is similar to that independently obtained by (D’Almeida, 1992), who demonstrated necessary and sufficient conditions for a curve to be an outline, but did not show that an outline determines a surface. Note that the theorem is trivial for surfaces of degree two, as generic surfaces of degree 2 are all projectively equivalent. The main result is given by the following two properties of outlines:

- The contour generator of a generic algebraic surface is determined as a *space curve* uniquely (up to a projectivity of space), by a generic projection of the curve on to a plane. In particular, the outline of an algebraic surface in a given view contains sufficient information to compute both the contour generator of that surface and the focal point through which the contour generator was formed, together in some arbitrary projective frame.
- Given the contour generator of a generic surface viewed from a generic focal point, and that focal point, the surface can be uniquely determined.

## 2. The outline of an algebraic surface

Throughout the paper, we assume an idealised pinhole camera. These cameras possess a *focal point* and an *image plane*, which is modelled as the projective plane. Points in space appear in the image as the intersection between the image plane and a line through the focal point and the point in space. An orthographic view occurs when the focal point is “at infinity”.

It is easy to see that if the focal point is fixed and the image plane is moved, the resulting distortion of the image is a *collineation*, a one-to-one map of the projective plane to itself that takes straight lines to straight lines. In what follows, it is assumed that neither the position of the image plane with respect to the focal point nor the size and aspect ratio of the pixels on the camera plane is known, so that the image presented to the algorithm is within some arbitrary collineation of the “correct” image. In this model, the image plane makes no contribution to the geometry, and its position in space is ignored.

The *outline* of a surface is a plane curve in the image, which itself is the projection of a space curve, known as a *contour generator*<sup>3</sup>. The contour generator is given by those points on the surface where the surface turns away from the image plane; formally, the line through the focal point to the surface is tangent to the surface. As a result, at an outline point, if the relevant surface patch is visible, nearby pixels in the image will see vastly different points on the surface, and so outline points usually have sharp changes in image brightness associated with them. Figure 1 illustrates these concepts.

We shall study generic algebraic surfaces in projective 3-space, otherwise written as  $P^3$ . A point in  $P^3$  is given by four homogenous coordinates, where two sets of homogenous coordinates refer to the same point if they are within a scalar multiple of one another. Appendix 1 in (Mundy and Zisserman, 1992b) contains examples and discussions of the practical applications of homogenous coordinates. Projective three-space is similar to the three dimensional space in which we live, but contains an extra plane, consisting of infinitely distant points. In  $P^3$ , an algebraic surface is given by the vanishing of a single homogenous polynomial in these four coordinates.

# Missing Figure

Fig. 1. The outline and contour generator of a curved object, viewed from a perspective camera.

We will assume that the camera has an infinite film plane as well, so that the image plane can be modelled by  $P^2$ , the projective plane. The points in space that project to points “at infinity” in the camera film plane lie on the plane parallel to the image plane and passing through the focal point.

We use the following notation:

- $(u_0, u_1, u_2, u_3)$  are the coordinates of a point in  $P^3$ .
- $(x_0, x_1, x_2)$  are the coordinates of a point in  $P^2$ , the image plane.
- $(f_0, f_1, f_2, f_3)$  is the camera focal point.
- $S(u_0, u_1, u_2, u_3)$  is the homogenous polynomial that vanishes on the surface.
- $d$  is the degree of  $S$ .

Since the surface is generic, it is irreducible; that is,  $S(u_0, u_1, u_2, u_3)$  does not factor. The contour generator lies on the surface, and so  $S(u_0, u_1, u_2, u_3) = 0$  on the contour generator. The plane tangent to the surface at a point on

the contour generator must pass through the focal point, by the definition of the contour generator. As a result, the expression

$$f_0 \frac{\partial S}{\partial u_0} + f_1 \frac{\partial S}{\partial u_1} + f_2 \frac{\partial S}{\partial u_2} + f_3 \frac{\partial S}{\partial u_3}$$

vanishes on the contour generator. This expression will be called  $T$  for short in what follows. We immediately have that:

- The contour generator is an algebraic space curve given by the vanishing of just two polynomials<sup>4</sup>,  $S$  and  $T$ , and so is a *complete intersection*. Furthermore, for a generic choice of the focal point,  $T$  has degree  $d - 1$ , and so the contour generator has degree  $d(d - 1)$ . The surface given by  $T = 0$  is known as the first polar of  $S$ .
- The *family* of contour generators on a surface is a family of curves *linearly* parametrised by focal points alone. Such families are known

to algebraic geometers as linear systems, and are widely studied. The study of such systems makes general statements about contour generators possible. For example, a generic contour generator on a generic surface is smooth, by Bertini's theorem<sup>5</sup>. In fact, the contour generator is a plane section of the dual of the surface, where the sectioning plane depends on the focal point chosen. The simplicity of the system of contour generators stands in stark contrast to the complexity displayed by the family of outlines, as the focal point changes, information conventionally captured by an aspect graph.

- Contour generators are projectively covariant; that is, for a surface  $S$ , viewed from a focal point  $f$ , with contour generator  $C$ , if  $P$  is an arbitrary projectivity of space, then  $P(C)$  is the contour generator of  $P(S)$  viewed from  $P(f)$ . This is because the contour generator is defined by tangency and incidence conditions alone.

It is important to note that the treatment that follows assumes that the complex points of both the algebraic curve and the algebraic surface are meaningful. For example, when a count is given of the number of singular points on the outline of a given type, that count includes the complex singularities. It is conceivable that an algebraic surface could have an outline that consisted entirely of complex points; a natural example is the outline of a sphere viewed from a focal point lying inside the sphere. In this case, while in principle the outline constrains the surface just as effectively as if it had a large collection of real points, in practice the outline is difficult to observe.

Another effect that can make the outline difficult to observe is self-occlusion, where sections of the outline are occluded by the surface and so are, in practice, invisible. Self occlusion is difficult to study given the methods here, which emphasize the global structure of curves and surfaces, and so cannot predict which section of an outline will be occluded. However, the outline of a generic algebraic surface is an irreducible algebraic curve (by genericity), meaning that a finite collection of points determines the complete geometry of the outline. This property means that partial occlusion is irrelevant for the cases studied here.

This is why the statement of the theorem emphasizes the equation of the outline. In practice,

if the outline has sufficient visible real points that a fitting process can determine its equation from the real points alone, its complex points and singularities follow. In principle, a fitting process should be robust to occlusions, as for irreducible algebraic curves (the genericity assumptions assure that the outlines covered in this paper are irreducible), only a finite number of points is necessary to determine the equation of the curve. This means that, to apply the result, we must assume that the view yields enough real points on the outline to determine its equation; this is not a particularly strong restriction in principle.

### 2.1. The singularities of the outline

Since the contour generator of a non-singular algebraic surface viewed through a generic focal point is smooth, the singularities of the outline must be a result of the projection from the contour generator to the outline. These singularities are the key to obtaining the contour generator from the outline; fortunately, they are highly structured. Generically there are only cusps and nodes. The following results have been known since at least the late 19th century; see, for example, (Basset, 1910).

**Cusps** A cusp in the outline is a local event on the contour generator, so that cusps are relatively easily studied; a cusp occurs when the contour generator is tangent to the ray through the focal point; see, for example, (Koenderink, 1992) for this widely known result.

**LEMMA 1** *Cusps in the outline are the projections of points on the contour generator where the second polar of the surface through the focal point vanishes; accordingly, there are  $d(d-1)(d-2)$  cusps in the outline of a surface of degree  $d$ .*

*Proof:* If  $p$  is a point on the contour generator that projects to a cusp on the outline, and  $f$  is the focal point, then the line  $pf$  is tangent to the contour generator at  $p$ . For any point  $q$  on the contour generator, the line tangent to the contour generator at  $q$  is given by the intersection of the plane tangent to the surface at  $q$  and the plane tangent to the first polar at  $q$ . Because the line

tangent at  $p$  passes through  $f$ , the plane tangent to the first polar at  $p$  must pass through  $f$  as well. Recall that the surface was written as  $S$  and the first polar was written as  $T$ . We have that the expression:

$$f_0 \frac{\partial T}{\partial u_0} + f_1 \frac{\partial T}{\partial u_1} + f_2 \frac{\partial T}{\partial u_2} + f_3 \frac{\partial T}{\partial u_3}$$

must vanish at  $p$ . This expression, which is the first polar of  $T$  through  $f$ , is also known as the second polar of  $S$  through  $f$ , and has degree  $d-2$  if  $d$  is the degree of  $S$ ; call this expression  $P$  for convenience.

In turn, if  $S$ ,  $T$  and  $P$  vanish at a point  $p$ , then the point is on the surface and on the contour generator by definition; furthermore, the plane tangent to the surface at  $p$  passes through  $f$ , and the plane tangent to  $T$  at  $p$  passes through  $f$ , so their intersection, which is tangent to the contour generator, passes through  $f$ . Thus, the contour generator cusps at exactly those points where  $S$ ,  $T$  and  $P$  vanish; by Bézout's theorem there are  $d(d-1)(d-2)$  such points, and so the outline has  $d(d-1)(d-2)$  cusps. ■

**Double points** Double points (nodes) on the outline occur when a line through the focal point is tangent to  $S$  in two distinct points, and so are global events; determining the number of double points on the outline requires more complex reasoning.

**LEMMA 2** *There are  $\frac{1}{2}d(d-1)(d-2)(d-3)$  double points on the outline of an algebraic surface of degree  $d$ .*

*Proof:* The contour generator is a complete intersection, and so its genus is given by the formula

$$g_{cg} = \frac{1}{2}d(d-1)(2d-5) + 1$$

where  $d$  is the degree of the surface (cf Hartshorne, 1977, p. 188, ex 8.4g). Project the contour generator into the image through the focal point; the resulting curve is birational to the contour generator, and so has the same genus. The singularities are stable (by the generic choice of surface and fo-

cal point), so the genus-degree formula for plane curves yields that

$$g_{cg} = (d^2 - d - 1)(d^2 - d - 2) - (n_c + n_d)$$

where  $n_c$  is the number of cusps and  $n_d$  is the number of double points. Rearranging the formula and substituting the above result on the number of cusps yields  $n_d = \frac{1}{2}d(d-1)(d-2)(d-3)$ . ■

## 2.2. Global properties of the singularities of the outline

A property of the outline that will prove important later is that its singularities lie on the intersection of two plane curves, whose degree (which is relatively low for the number of points) can be determined using elimination theory. These curves can be studied, without loss of generality, by assuming that the focal point is the point  $(0, 0, 0, 1)$ . This makes computing the outline relatively simple; a point  $(u_0, u_1, u_2, u_3)$  projects through  $(0, 0, 0, 1)$  to  $(u_0, u_1, u_2)$ , because if  $(u_0, u_1, u_2)$  is fixed and the fourth coordinate varies, the locus of points obtained is a line, limiting to the origin as  $u_3$  becomes large. Thus,  $(u_0, u_1, u_2)$  yield the line, and  $u_3$  is a coordinate along the line.

The equation of a surface  $S$  of degree  $d$  can be rewritten as:

$$S(u_0, u_1, u_2, u_3) = H_0(u_0, u_1, u_2)u_3^d + H_1(u_0, u_1, u_2)u_3^{d-1} + \dots + H_d(u_0, u_1, u_2)$$

where  $H_i(u_0, u_1, u_2)$  is homogenous of degree  $i$  in  $u_0$ ,  $u_1$ , and  $u_2$ . The focal point is  $(0, 0, 0, 1)$ , so that the first polar through the focal point is:

$$\frac{\partial S}{\partial u_3} = dH_0(u_0, u_1, u_2)u_3^{d-1} + (d-1)H_1(u_0, u_1, u_2)u_3^{d-2} + \dots + H_{d-1}(u_0, u_1, u_2)$$

and this vanishes on the contour generator too. Now the outline consists of those points  $(u_0, u_1, u_2)$  where both equations vanish; the equation of the outline is therefore obtained by eliminating  $u_3$  between  $S$  and  $\frac{\partial S}{\partial u_3}$ . The singularities of the outline all have multiplicity two, and are those points  $(u_0, u_1, u_2)$  where  $S$  and  $\frac{\partial S}{\partial u_3}$  have

two common roots in  $u_3$ . The equations yielding these points can be obtained using a technique from (Salmon).

Consider two polynomials in  $u_3$ ,

$$F(u_3) = \sum_{i=0}^{i=d} a_{d-i} u_3^i$$

and

$$G(u_3) = \sum_{i=0}^{i=d-1} b_{d-1-i} u_3^i$$

If  $F$  and  $G$  have two common roots, then there must be some

$$M(u_3) = \sum_{i=0}^{i=d-3} A_{d-3-i} u_3^i$$

and

$$N(u_3) = \sum_{i=0}^{i=d-2} A_{2d-4-i} u_3^i$$

such that  $FM + GN = 0$  identically.  $M$  is the product of all the factors of  $G$  that do not appear in  $F$ , and  $N$  is the product of all the factors of  $F$  that do not appear in  $G$ . The polynomial  $FM + GN$  has degree  $2d - 3$ , and there are  $2d - 3$  unknown  $A_i$ , and  $2d - 2$  monomials in  $u_3$ . We can construct a  $2d - 3$  by  $2d - 2$  matrix  $\mathbf{C}$ , such that  $FM + GN = \mathbf{a}^t \mathbf{C} \mathbf{u}$ , where  $\mathbf{a}$  is the vector

$$(A_0, A_1, \dots, A_{d-3}, A_{d-2}, A_{d-1}, \dots, A_{2d-5}, A_{2d-4})^t$$

$\mathbf{u}$  is the vector

$$(u_3^{2d-3}, u_3^{2d-2}, \dots, u_3^2, u_3, 1)$$

and  $\mathbf{C}$  is the  $2d - 3$  by  $2d - 2$  matrix whose entries are shown in figure 2.

Since, for an appropriate choice of  $A_i$ ,  $FM + GN = 0$  identically (i.e. all the coefficients vanish), there is some choice of  $\mathbf{a}$  such that  $\mathbf{a}^t \mathbf{C} = 0$ . Hence, the  $2d - 3$  by  $2d - 3$  minors of  $\mathbf{C}$  must vanish.

In our case,  $a_j = H_j$ , and  $b_j = (d - j)H_j$ . The method of construction of the matrix ensures that the minors are homogenous; the degree of a minor in  $(u_0, u_1, u_2)$  can be determined by computing the degree of a typical monomial in the minor. Such

a monomial can be obtained by striking one column of  $\mathbf{C}$ , and multiplying  $2d - 3$  elements from the remaining square matrix using each row and each column only once. The resulting monomial will have the form  $H_a H_b H_c \dots$ , and its degree is the sum of the subscripts. This process shows the degrees of the minors are:

$$(d-1)(d-2), (d-1)(d-2) + 1, \\ (d-1)(d-2) + 2, \dots, (d-2)(d-1) + 2d - 3$$

At a singularity of the outline, these minors must all vanish, so that there exists a family of curves, which intersect at most in points, of these degrees, which pass through the singular points. In particular, the singularities must lie on (though not necessarily exhaust) the intersection of a curve of degree  $(d-1)(d-2)$  with a curve of degree  $(d-1)(d-2) + 1$ , where these curves do not have a common component.

This means that the singularities are strongly constrained. A curve of degree  $s$  has  $(1/2)(s+1)(s+2)$  coefficients, meaning that  $(1/2)(s+1)(s+2) - 1$  general points uniquely specify such a curve. There are in total  $(1/2)(d^2 - 2d)(d^2 - 2d + 1)$  singularities; if these were in general position, the lowest degree curve that would pass through all of them would have degree  $(d-1)^2$ .

The matrix  $\mathbf{C}$  yields a great deal of information about the structure of the problem. Write

$$\mathbf{C} = (\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{2d-3})$$

where the  $\mathbf{c}_i$  are column vectors. Let

$$\mathbf{C}_1 = (\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{2d-4})$$

By inspecting the diagonal elements, it can be seen that the determinant of  $\mathbf{C}_1$ , which is square, has degree  $(d-1)(d-2)$ . Let  $\mathbf{D} = \text{Adjoint}(\mathbf{C}_1)$  (where the adjoint is the transpose of the matrix of co-factors), and let

$$\mathbf{C}_r = (\mathbf{c}_0, \mathbf{c}_1, \dots, \mathbf{c}_{2d-6}, \mathbf{c}_{2d-5}, \mathbf{c}_{2d-3})$$

Inspecting the diagonal elements shows that  $\text{Det}(\mathbf{C}_r)$  has degree  $(d-1)(d-2) + 1$ . Write  $\mathcal{P}$  for  $\text{Det}(\mathbf{C}_1)$  and  $\mathcal{Q}$  for  $\text{Det}(\mathbf{C}_r)$ . Now both  $\mathcal{P}$  and  $\mathcal{Q}$  are  $2d - 3$  by  $2d - 3$  minors of  $\mathbf{C}$ , and so must vanish on all the singular points of the outline.

Since  $\mathbf{C} \mathbf{u}$  is a vector of polynomials, all of which vanish at every point on the contour generator,

$$\begin{array}{cccccccccccc}
a_0 & a_1 & a_2 & \dots & \dots & \dots & a_d & 0 & 0 & 0 & \dots & 0 \\
0 & a_0 & a_1 & a_2 & \dots & \dots & \dots & a_d & 0 & 0 & \dots & 0 \\
0 & 0 & a_0 & a_1 & a_2 & \dots & \dots & \dots & a_d & 0 & \dots & 0 \\
\dots & \text{(d-2 rows)} & & & & & & & & & & \\
0 & 0 & 0 & \dots & \dots & a_0 & a_1 & a_2 & \dots & \dots & \dots & a_d \\
b_0 & b_1 & \dots & \dots & \dots & b_{d-3} & b_{d-2} & b_{d-1} & 0 & 0 & \dots & 0 \\
0 & b_0 & b_1 & \dots & \dots & \dots & b_{d-3} & b_{d-2} & b_{d-1} & 0 & \dots & 0 \\
\dots & \text{(d-1 rows)} & & & & & & & & & & \\
0 & 0 & 0 & 0 & \dots & b_0 & b_1 & \dots & \dots & b_{d-3} & b_{d-2} & b_{d-1}
\end{array}$$

Fig. 2. The entries in the  $2d-3$  by  $2d-2$  matrix  $\mathbf{C}$ , whose minors are the polynomials that vanish on the singularities of the outline.

$\mathbf{DCu}$  must also consist of a vector of polynomials, each of which vanishes at every point on the contour generator. In particular, the last row of  $\mathbf{DC}$  has the form:

$$(0, 0, 0, \dots, 0, \mathcal{P}, \mathcal{Q})$$

and so the last element of  $\mathbf{DCu}$  is the equation

$$\mathcal{P}u_3 + \mathcal{Q}$$

which must vanish at every point on the contour generator. This equation cannot vanish trivially; that is, at “almost every” point on the contour generator, both  $\mathcal{P}$  and  $\mathcal{Q}$  are non-zero, by the following argument:

both  $\mathcal{P}$  and  $\mathcal{Q}$  are homogenous polynomials in the variables  $u_0$ ,  $u_1$  and  $u_2$  and so they vanish on a cone passing through the point  $(0, 0, 0, 1)$ . If  $\mathcal{P}$  and  $\mathcal{Q}$  were to vanish on the entire contour generator, this cone would contain the contour generator, and so  $\mathcal{P}$  and  $\mathcal{Q}$  would have to vanish on the projection of the contour generator through the point  $(0, 0, 0, 1)$  to any plane. However, a projection of the contour generator to a plane through  $(0, 0, 0, 1)$  must (by the generic choice of surface) be irreducible and have degree  $d(d-1)$ ; neither  $\mathcal{P}$  nor  $\mathcal{Q}$  can vanish at every point of an irreducible curve of this degree, because their degrees are too low.

This means that, if  $\mathcal{P}$  and  $\mathcal{Q}$  can be determined from the image, the contour generator can be reconstructed from the outline, because the “missing” homogenous coordinate of the contour generator,  $u_3$  (which can loosely be thought of as “depth”) can be determined as

$$u_3 = \frac{-\mathcal{Q}}{\mathcal{P}}$$

This expression, though not strictly a function, is meaningful, because the degree of  $\mathcal{Q}$  is one larger than the degree of  $\mathcal{P}$ ; as a result, if  $(u_0, u_1, u_2)$  were to be scaled by  $\lambda$ , the expression for  $u_3$  would be scaled by  $\lambda$  too.

In fact,  $\mathcal{P}$  and  $\mathcal{Q}$  can be determined from image information alone, up to an ambiguity which is a subgroup of the projective group;  $\mathcal{P}$  is the only polynomial of degree  $(d-1)(d-2)$  that vanishes on all the singularities of the outline, and hence can be determined from image information up to scale. In turn,  $\mathcal{Q}$  is a polynomial of degree  $(d-1)(d-2)+1$  that vanishes on all the singularities of the outline. There is a four dimensional space of such polynomials; section 3.1 shows that the ambiguity arising from choosing one of these polynomials to act as  $\mathcal{Q}$  arbitrarily is just a projective transformation of the contour generator.

The constraints that singularities lie on curves of particular degrees determine the family of curves that are generic outlines of smooth surfaces, according to an result of (D’Almeida, 1992) which states that:

**Theorem:** (D’Almeida) *Let  $\Gamma$  be a plane curve of degree  $n(n-1)$ ,  $n \geq 3$ . The necessary and sufficient condition that there exists a smooth surface  $S \subset P^3$  and a generic point  $p$  of  $P^3$  such that  $\Gamma$  is the curve of ramification of the projection of  $S$  through  $p$ , is as follows:*

*The curve  $\Gamma$  has  $d = n(n-1)(n-2)(n-3)/2$  ordinary double points,  $k = n(n-1)(n-2)$  cusps and no other singularities. There are two curves  $\mu_0$  and  $\mu_1$  of degrees  $n^2 - 3n + 2$*

and  $n^2 - 3n + 3$  respectively, without a common component, that pass through the singular points of  $\Gamma$ . The minimal degree of a plane curve containing the singular points of  $\Gamma$  is  $n^2 - 3n + 2$

Note that the ‘‘curve of ramification’’ is equivalent to our outline. Any errors in translation are mine.

### 2.3. Further global properties of the outline

The study of outlines is quite rich in curious geometric properties; in particular, the form of a generic outline is strongly constrained, and the projective invariants of a generic outline must satisfy constraints. For example, note that there must exist a frame in which the outline of a cubic surface has the form  $C^2 - Q^3 = 0$ , where  $Q$  is quadratic and  $C$  is cubic.

This can be shown by representing the equation of surface as

$$H_0(u_0, u_1, u_2)u_3^3 + H_1(u_0, u_1, u_2)u_3^2 + H_2(u_0, u_1, u_2)u_3 + H_3(u_0, u_1, u_2)$$

By choice of frame, the focal point can be given coordinates  $(0, 0, 0, 1)$  and we can ensure  $H_1(u_0, u_1, u_2) = 0$  identically; divide by  $H_0$  (which is a constant), to get the form

$$u_3^3 + H_2(u_0, u_1, u_2)u_3 + H_3(u_0, u_1, u_2)$$

The polar through the focal point is now

$$T(u_0, u_1, u_2, u_3) = 3u_3^2 + H_2(u_0, u_1, u_2)$$

The resultant with respect to  $u_3$  has degree six, and consists of terms formed from  $H_3$  (degree 3) and  $H_2$  (degree 2), and so must have the form  $C^2 - Q^3 = 0$ , for an appropriate choice of  $C$  and  $Q$ .

Similar statements are possible about the outlines of surfaces of higher degree, but the form of the constraint becomes more complex; a possible benefit of such a result includes controlling the complexity of the fitting problem - most algebraic curves are not outlines.

## 3. Obtaining the contour generator from the outline

Determining the contour generator from the outline requires knowledge of the ‘‘depth’’ to the contour generator at each point of the outline. The last sections indicated how this depth is to be found, by showing an expression that gives the homogenous coordinate  $u_3$  as a rational function of the other three homogenous coordinates on the outline. In particular, this rational function can be determined from the singularities of the outline using the property that both numerator and denominator vanish on the singularities of the outline. This means that the expression for  $u_3$  is undetermined at these points. Surprisingly, this is a useful property, because it makes it possible to obtain a non-singular space curve from a singular plane curve. In particular, the process sketched above for determining the contour generator from the outline is widespread in algebraic geometry, and is known as ‘‘blowing up.’’ This section provides some simpler examples of blowing up to demonstrate how the process can ‘‘undo’’ singularities; it then shows that the reconstruction of the contour generator is correct by showing that it is the only possible such reconstruction, up to a projective transformation of space. This latter result requires some complicated machinery, which is briefly introduced.

### 3.1. Blowing up

The outline has only cusps and double points as singularities, by the assumption that both surface and viewing position are generic. This means that there is no need to blow up more complex singularities. In the case of blowing up cusps or double points, the central issue is a depth function that can be evaluated along the plane span of the curve, giving the coordinates of the space curve in affine coordinates as

$$\left(x, y, \frac{f(x, y)}{g(x, y)}\right)$$

or in homogenous coordinates as

$$\left(x_0, x_1, x_2, \frac{F(x_0, x_1, x_2)}{G(x_0, x_1, x_2)}\right)$$

This is equivalent to:

$$(G(x_0, x_1, x_2)x_0, G(x_0, x_1, x_2)x_1, G(x_0, x_1, x_2)x_2, F(x_0, x_1, x_2))$$

Clearly, in the case of homogenous coordinates the degree of  $G$  is one less than the degree of  $F$ . The depth function must have two values at each double point (these are given as limits as a point on the curve approaches the double point), so as to construct two points at different depths in space that correspond to the single point in the image. As the following two examples show, this is achieved by having the depth function undefined ( $0/0$ ) at the singularities, with appropriate limiting properties close to the singularities; in the case of homogenous coordinates, all four homogenous coordinates vanish simultaneously, again with appropriate limiting properties.

**Example: blowing up a double point in the affine plane:** Consider the curve given by  $y^3 - x^2 + y^2 = 0$ , which has a double point at the origin where the curve crosses itself transversally. The curve can be parametrised as

$$(x, y) = (t^3 - t, t^2 - 1)$$

where  $t$  is some complex parameter. The curve passes through the double point when  $t = 1$  or  $t = -1$ .

The function

$$f(x, y) = (x, y, x/y)$$

which takes a point in the plane to a point in space, is undefined at the origin; furthermore,

$$\lim_{t \rightarrow 0} f(t \cos \theta, t \sin \theta)$$

depends on  $\theta$ , so that when the function is applied to a curve approaching the origin, the value of the  $z$ -coordinate depends on the direction of the approach.

In particular, applying this function to the curve under consideration produces

$$(x, y, z) = (t^3 - t^2, t^2 - 1, t)$$

less the points  $t = 1$  and  $t = -1$ , where the function is not defined. However, at these points the space curve has meaningful limits, which are

$(0, 0, 1)$  and  $(0, 0, -1)$ . By attaching these limit points we obtain a smooth space curve from a singular plane curve.

**Example: blowing up a cusp in the projective plane:** In the projective plane, points are given by three homogenous coordinates. In this case, a polynomial cannot be a function, because scaling each homogenous coordinate changes the value of the polynomial without changing the point at which the function is defined. Thus, functions are given by ratios of homogenous polynomials of the same degree in homogenous coordinates. In fact, a function that maps a curve in the projective plane to a curve in *projective* space can be given as *four* homogenous polynomials of the same degree in the homogenous coordinates of the plane; in this form, each polynomial represents a homogenous coordinate in space.

Consider the curve given by  $x_0^3 - x_2x_1^2 = 0$  in the projective plane; this curve has a cusp at  $(0, 0, 1)$ , and can be parametrised as  $(r^2s, r^3, s^3)$ , where  $(r, s)$  are the homogenous coordinates of a point on the projective line. Consider the following function from the projective plane to projective three-space:

$$f(x_0, x_1, x_2) = (x_0^2, x_1x_0, x_1x_2, x_0x_2)$$

Applied to the curve, this function yields the parametric space curve given in homogenous coordinates by:

$$(r^4s^2, r^5s, r^3s^3, r^2s^4)$$

which is equivalent to that given in homogenous coordinates by:

$$(r^2s, r^3, rs^2, s^3)$$

This curve is a twisted cubic - this is perhaps easiest to see by dividing by the fourth coordinate, writing  $r/s = t$  and ignoring the point at infinity, giving the curve in affine (non-homogenous) coordinates as  $(t^2, t^3, t)$ ; this space curve has no singularities.

**Blowing up the outline** The key to blowing up a curve with double points and cusps, as the examples have shown, is to obtain a depth function that goes to  $0/0$  at the double points and cusps of the curve. For the outline of an algebraic

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Fig. 3. Four frames from a fly-by, showing a plane curve with a double point and its blow-up. The blown-up curve is a non-singular space curve, shown here lying above the plane curve. It projects to the plane curve under orthographic projection in this case.

curve in affine coordinates, such a function is easily available. Recall from section 2.2 that there exist two equations  $\mathcal{P}$  of degree  $(d - 1)(d - 2)$ , and  $\mathcal{Q}$  of degree  $(d - 1)(d - 2) + 1$  (with no factor in common with  $\mathcal{P}$ ), both of which vanish on the singularities of the outline. These equations yield the necessary depth function.

Because the reconstruction is proceeding up to a projective ambiguity, it is possible to choose a focal point; choose this focal point to be  $(0, 0, 0, 1)$ , to simplify the working. Now the contour generator is some curve  $(u_0(t), u_1(t), u_2(t), u_3(t))$ , and the outline in the image plane consists of the curve  $(x_0(t), x_1(t), x_2(t)) = (u_0(t), u_1(t), u_2(t))$ . Reconstructing the contour generator consists, in effect, of supplying the missing  $u_3(t)$ . However, from the previous section,  $\mathcal{P}u_3 + \mathcal{Q} = 0$  on the contour generator, and  $\mathcal{P}$  and  $\mathcal{Q}$  are expressions in  $u_0, u_1$ , and  $u_2$  alone, which can be determined from the image information, so that  $u_3$  can be determined at each point on the curve.

In particular, given an outline in the projective plane, apply the map

$$(x_0, x_1, x_2) \rightarrow (x_0\mathcal{P}, x_1\mathcal{P}, x_2\mathcal{P}, \mathcal{Q})$$

taking every point on the outline to a point in space. For convenience, call this map the “lifting map”. At the singular points of the outline, the lifting map degenerates (as both  $\mathcal{P}$  and  $\mathcal{Q}$  vanish at these points, the image of these points in the map given is  $(0, 0, 0, 0)$ , which is not a meaningful point in projective space). The result of the following section shows that the closure (required to fill in the missing points where the map degenerates) of the image of the outline in the lifting map must be the contour generator.

The lifting map has further useful properties; in particular, it has the property that

$$\pi \circ Lift = Identity$$

where  $\pi$  is projection through the point  $(0, 0, 0, 1)$ . In coordinates, drop the fourth homogenous coordinate, so that

$$\pi \circ Lift : (x_0, x_1, x_2) \rightarrow (x_0\mathcal{P}, x_1\mathcal{P}, x_2\mathcal{P})$$

Since we are working in homogenous coordinates,  $(x_0\mathcal{P}, x_1\mathcal{P}, x_2\mathcal{P})$  is equivalent to  $(x_0, x_1, x_2)$ . This means that, if *Lift* takes the outline to the contour generator, it does so with a notion of the appropriate focal point through which to project the contour generator back on to the outline - the particular lift constructed presumes that the focal point is  $(0, 0, 0, 1)$ , which can be done without loss of generality by choice of coordinates. Any other particular focal point can be chosen as well, though the form of the resulting lift is slightly more complicated; the important thing is that, once the lifting process has been applied, both the contour generator and the focal point are available, in a single coordinate system. This data is sufficient to determine the surface.

The lifting map contains an intrinsic projective ambiguity, because  $\mathcal{Q}$  cannot be determined uniquely. There are sufficient singularities for  $\mathcal{P}$  to be known up to a scale - which is not a source of ambiguity, because we are working in homogenous coordinates - but there is a four-dimensional space of curves of degree  $(d - 1)(d - 2) + 1$  that vanish on the singularities, spanned by  $(\mathcal{Q}, x_0\mathcal{P}, x_1\mathcal{P}, x_2\mathcal{P})$ . An element of this space is given by  $\mathcal{Q}_a = a_0x_0\mathcal{P} + a_1x_1\mathcal{P} + a_2x_2\mathcal{P} + a_3\mathcal{Q}$ . Now if the lifting map uses  $\mathcal{Q}_a$  instead of  $\mathcal{Q}$ , the resulting curve is:

$$(x_0\mathcal{P}, x_1\mathcal{P}, x_2\mathcal{P}, \mathcal{Q}_a) = (x_0\mathcal{P}, x_1\mathcal{P}, x_2\mathcal{P}, \mathcal{Q})\mathbf{M}$$

where  $\mathbf{M}$  is the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & a_0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & a_3 \end{pmatrix}$$

This is clearly just a projective transformation, as long as  $a_3 \neq 0$ . Since both  $\mathcal{P}$  and the whole space of possible  $\mathcal{Q}_a$ 's can be determined from the outline, satisfying the requirement that  $a_3 \neq 0$  simply involves choosing a  $\mathcal{Q}_a$  that does not share a factor with  $\mathcal{P}$ , which is easily done.

### 3.2. Uniqueness of the lift

The sections above have shown constructively that the outline can be lifted to yield the contour generator, and have demonstrated a lifting process that must yield the contour generator from the outline. It is also possible to show that this is, up to projective transformations, the only process that will do so; the proof is not novel, and requires a certain amount of technical algebraic geometry, but is included here for completeness. Space does not allow a comprehensive introduction to the material required, but subsection 3 introduces the general approach, and sketches the direction that the mathematics in subsection 3 takes, as the form of argument used represents a powerful tool for solving questions about space curves. The reader is referred to (Hartshorne, 1977), which is difficult but comprehensive, or to (Gomez-mont, 1989), which is much more approachable but less wide-ranging. The reader willing to accept that the lifting process in the previous section yields the contour generator may wish to skip both sections, or read only subsection 3.

**Thrust of the mathematics** The central question is: given a projection of an abstract algebraic curve satisfying particular constraints, in how many projectively different ways could that curve be embedded in space, consistent with the image data? The result that will appear is that there is a natural choice of depth function to obtain the contour generator from the outline. This

result is a statement about the possible embeddings of a curve in space that are consistent with the image data.

Embeddings of curves are generally attacked through a technical device called a *line bundle*, which consists of a collection of sets made up of the cartesian product of an open set on the curve and an affine line. These sets are pasted together in a precise way using *transition functions*. Transition functions are associated with the intersection of two of these sets; their domain is the open set on the curve, and their range is the line. Transition functions allow studies of *sections* of line bundles, which associate points on the line with points on the curve. Formally, a section is a map from the curve to the line bundle, so that the projection of the map onto the first factor is the identity; this means that, in some coordinate system, the map has the form  $f : p \rightarrow (p, q)$ , where  $p$  is a point on the curve and  $q$  is a point on the line. Where two sets intersect, there are two ways of writing each point on the curve and each point on the line - one set of coordinates for each set. Transition functions define the correspondence between points on the line in the coordinates associated with the first set, and those in the coordinates associated with the second set. In fact, the choice of transition functions yields the bundle.

The result is an object that locally looks like a piece of curve crossed with the affine line (c.f. the vector bundles of differential geometry). Line bundles in algebraic geometry have more rigidity properties than the bundles of differential geometry, for two reasons. Firstly, the topology used to define open sets is the Zariski topology, where all algebraic sets are closed; this means that an open set on a curve consists of the whole curve, less some finite number of points. Secondly, the bundles under consideration are typically *holomorphic* - this means that the transition functions are analytic in their domain.

The following example, which is a modified version of example 4.7 in (Gomez-mont, 1989), displays a family of line bundles over the projective line. The first open set on the projective line will consist of the points given in coordinates as  $s$ , for  $s$  some complex number (henceforth, the complex numbers will be denoted by  $\mathcal{C}$ ); call this set  $U$ . This is the projective line less one point, the point

at infinity. The second open set will consist of the points given in coordinates as  $t$ , for  $t$  some complex parameter; call this set  $V$ . Again, this is the projective line less a point (which would be the origin in  $s$  coordinates). Define the change of coordinates in crossing from  $U$  to  $V$  by  $t = 1/s$ . The two sets, pasted together in this way, give the whole of the projective line; figure 4 illustrates how the sets are assembled together to yield a line.

The line bundle will consist of the sets  $U \times \mathcal{C}$  and  $V \times \mathcal{C}$ , pasted together in an appropriate way. The set  $U \cap V$  consists of the whole line, less two points. There must be two transition functions:  $f_{VU}$ , which takes coordinates on the line in  $U$ 's frame to those in  $V$ 's frame, and  $f_{UV}$ , which takes coordinates on the line in  $V$ 's frame to those in  $U$ 's frame. Consider a point  $p$  in  $U \cap V$ . Write:

- $p_U$  for the coordinates of  $p$  in  $U$ 's frame;
- $q_U$  for the coordinate in  $U$ 's frame of a point on the line  $\mathcal{C}$ ;
- $p_V$  for the coordinates of  $p$  in  $V$ 's frame;
- $q_V$  for the coordinate in  $V$ 's frame of the point on the line  $\mathcal{C}$  that would be written  $q_U$  in  $U$ 's frame.

Then the pair  $(p_U, q_U)$  corresponds to the pair  $(p_V, q_V) = (p_V, f_{VU}(p_U)q_U)$ . Clearly, we have that  $f_{UV}f_{VU} = 1$ , and that neither function vanishes on  $U \cap V$ . We can now define a family of line bundles by the transition functions  $f_{VU} = s^{-n} = t^n$  and  $f_{UV} = s^n = t^{-n}$ . These functions specify how the coordinates of a holomorphic section change as we move from  $U$  to  $V$  and back.

In  $U$ , a holomorphic section of this bundle must have the form  $(s, \sigma(s))$ , where  $\sigma$  is a holomorphic function on  $\mathcal{C}$ . As a result,  $\sigma$  has a representation of the form

$$\sum_{i=0}^{i=\infty} a_i s^i$$

on  $U$ . In  $U \cap V$ , this section must also have the representation (in the coordinate  $t$  on  $V$ )

$$t^n \sum_{i=0}^{i=\infty} a_i t^{-i} = \sum_{i=0}^{i=\infty} a_i t^{n-i}$$

and this expression must also be holomorphic. In turn, this means that for  $n < 0$ , there are no holomorphic sections. For  $n \geq 0$ , there are holomor-

phic sections which have the form  $(s, \sum_{i=0}^{i=n} a_i s^i)$  (for any choice of  $a_i$ ) in  $U$  and in  $V$ , the form  $(t, \sum_{i=0}^{i=n} a_i s^{n-i})$ . Hence, for a given  $n$ , there is an  $n + 1$  dimensional vector space of holomorphic sections, corresponding to the choice of  $a_i$ .

Taken on its own, a section of a line bundle has no interest for us here; however, a ratio of holomorphic sections of a line bundle is a meromorphic (rational, with poles) function on the curve. Thus, in the example above, with  $n = 3$ , there is a four dimensional vector space of sections. The four sections given in  $U$  coordinates by  $(s, 1)$ ,  $(s, s)$ ,  $(s, s^2)$ ,  $(s, s^3)$  can be thought of as a map, applied to the curve, taking it to the points given in homogenous coordinates in space as:

$$(1, s, s^2, s^3)$$

These four distinct holomorphic sections of this line bundle map the line to the twisted cubic in projective space, less one point; this missing point is the image of the point at infinity, and can be obtained by evaluating these sections at the one point in  $V$  that does not lie in  $U \cap V$ . In general, four distinct holomorphic sections of a line bundle on a curve represent a map taking the curve to a curve in  $P^3$ , and the resulting space curve is algebraic. Note that sections can be added to one another or multiplied by constants, so that one usually considers the linear span of a set of sections. The attractive features of line bundles as tools are illustrated by our example:

- there are “few” holomorphic sections;
- it is very often possible to tell “how many” holomorphic sections there are;
- a bundle that has  $n$  independent holomorphic sections represents a map taking the curve to a curve in  $P^{(n-1)}$ .

As a result of these properties, line bundles are a central tool in studying embeddings of algebraic curves.

It can be seen from the example that different line bundles represent embeddings with different properties; we shall be concerned with a line bundle often represented as  $\mathcal{O}_C(1)$ , where  $C$  is the contour generator. In the case of the projective line given above, this would be the bundle that would result for  $n = 1$ . For a general plane curve  $C$ , a general section of  $\mathcal{O}_C(1)$  would vanish either on a set of points where a line intersects the

# Missing Figure

*Fig. 4.* Pasting together two affine lines to get a projective line. The first copy of the affine line parametrizes all the points on the projective line save the point at infinity; the second copy parametrizes the point at infinity, but lacks the origin. These two copies intersect almost everywhere; they are pasted together by specifying how the coordinate of a given point on one set relates to the coordinate of the equivalent point on the other.

curve, or on a set of points that are functionally equivalent to a linear section. In this case, functional equivalence means that the points are given by the vanishing of  $(p_1\pi)/p_2$ , where  $p_1$  and  $p_2$  are homogenous polynomials of the same degree,  $\pi$  is the equation of a line, and the expression  $(p_1\pi)/p_2$  has no poles, so that the zeros of  $p_2$  must all lie on zeros of  $p_1\pi$ . Clearly, if  $p_1$  is the equation of some arbitrary line and  $p_2 = \pi$ , this condition is satisfied. For some curves, there are other cases that will satisfy this condition. For example, if  $C$  is the outline of a surface, then  $p_1 = \mathcal{Q}$ ,  $p_2 = \mathcal{P}\pi$  will also satisfy this condition, where  $\mathcal{P}$ ,  $\mathcal{Q}$  are the equations vanishing on the singularities and defined above. This follows because the expression  $\mathcal{P}u_3 + \mathcal{Q}$ , which was shown above to vanish on the contour generator, demonstrates that all the zeros of  $\mathcal{P}$  that lie on the contour generator coincide with zeros of  $\mathcal{Q}$ .

For a space curve, a choice of four linearly independent sections of the bundle  $\mathcal{O}_C(1)$  gives an embedding of the curve in space; in particular, this bundle admits four sections that can be represented in coordinates as  $(u_0, u_1, u_2, u_3)$  (which basically just embeds the curve where it is in space). Four linearly independent sections chosen from the linear span of this family would yield an

embedding of the curve that is projectively equivalent to the original curve. More interestingly, three linearly independent sections chosen from the linear span of this family would represent a projection of this curve onto a plane through some focal point; to recover the space curve, one would need to determine a fourth section in the family generated as the span of  $(u_0, u_1, u_2, u_3)$  (which would generate our “depth function”). Of course, if  $\mathcal{O}_C(1)$  admits more than this four dimensional vector space of sections, the problem is hopeless, as it would not be possible to determine whether the fourth section chosen actually lies in the span of  $(u_0, u_1, u_2, u_3)$ , and so one could not know without other sources of information whether the embedding chosen corresponded to the correct one. The crucial fact is that  $\mathcal{O}_C(1)$  has only a four dimensional family of sections for  $C$  a contour generator (in fact, for  $C$  a complete intersection). This means that the fourth section can be determined from a projection of the curve up to at worst a projective ambiguity, so that the contour generator can be recovered from the outline.

**Mathematical details** At issue is  $\mathcal{O}_C(1)$ , for  $C$  the contour generator; if  $H^0(C, \mathcal{O}_C(1))$ , which is the space of sections of  $\mathcal{O}_C(1)$ , is isomorphic to

$H^0(P^3, \mathcal{O}_{P^3}(1))$ , then  $H^0(C, \mathcal{O}_C(1))$  has dimension four. Since the outline is birational to the contour generator,  $\mathcal{O}_O(1)$  is the same as  $\mathcal{O}_C(1)$ , where  $O$  represents the outline; three linearly independent sections of  $\mathcal{O}_O(1)$  are known (in coordinates,  $(x_0, x_1, x_2)$ ). If a fourth can then be determined, then  $O$  can be embedded in space using these four sections, and the result must be projectively equivalent to  $C$ .

**LEMMA 3** *Given an algebraic curve  $C$ , which is a complete intersection in  $P^3$  and is not a plane curve,  $H^0(C, \mathcal{O}_C(1))$  is isomorphic to  $H^0(P^3, \mathcal{O}_{P^3}(1))$ .*

*Proof:* I am indebted to Prof. O. DeBarré, of the Mathematics Department, University of Iowa, for pointing out this lemma, and showing me how it could be proven. This proof largely follows his; errors or inaccuracies are of my own addition. Note that a similar fact appears as an exercise in (Hartshorne, 1977) (p. 188, ex. 8.4).

Consider the following exact sequence of sheaves associated with the curve:

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{P^3} \rightarrow \mathcal{O}_C \rightarrow 0$$

where the symbols have their usual meaning as in, for example, (Hartshorne, 1977). Taking the associated cohomology sequence, and twisting by 1, we obtain the following long exact sequence:

$$0 \rightarrow H^0(P^3, \mathcal{I}(1)) \rightarrow H^0(P^3, \mathcal{O}_{P^3}(1)) \rightarrow H^0(C, \mathcal{O}_C(1)) \rightarrow H^1(P^3, \mathcal{I}(1)) \rightarrow \dots$$

$H^0(P^3, \mathcal{I}(1))$  represents those homogenous linear expressions that vanish on the curve, and must be empty because the curve does not lie in any plane.  $H^0(P^3, \mathcal{O}_{P^3}(1))$  represents the hyperplanes in  $P^3$  and  $H^0(C, \mathcal{O}_C(1))$  represents the space of sections of the line bundle given by a hyperplane section of  $C$ . If we can prove that  $H^1(P^3, \mathcal{I}(1))$  is empty, we have that  $H^0(C, \mathcal{O}_C(1))$  is isomorphic to the system of hyperplanes in  $P^3$ , and so that the sections of this bundle form a four-dimensional space.

The curve is a complete intersection, given (say) by  $p = 0, q = 0$ , for polynomials  $p$  and  $q$ . As a result, we have the following free resolution of its ideal:

$$0 \rightarrow R \rightarrow R \oplus R \rightarrow I \rightarrow 0$$

where  $R$  is the graded ring of homogenous polynomials in four variables over the complex numbers, and  $I$  is the curve's ideal. In this sequence, the injection  $R \rightarrow R \oplus R$  is given by  $f \rightarrow (-pf, qf)$ , and the surjection  $R \oplus R \rightarrow I$  is given by  $(a, b) \rightarrow qa + pb$ . Keeping track of the grading, we find:

$$0 \rightarrow R(1 - m - n) \rightarrow R(1 - m) \oplus R(1 - n) \rightarrow I(1) \rightarrow 0$$

This free resolution yields the exact sequence of line bundles:

$$0 \rightarrow \mathcal{O}_{P^3}(1 - m - n) \rightarrow \mathcal{O}_{P^3}(1 - m) \oplus \mathcal{O}_{P^3}(1 - n) \rightarrow \mathcal{I}(1) \rightarrow 0$$

Taking the associated cohomology sequence, and recalling the standard result (for example, in Hartshorne, 1977, p. 225) that

$$H^i(P^n, \mathcal{O}_{P^n}(j)) = 0$$

for  $0 < i < n$  and for all  $j \in Z$  gives that  $H^1(P^3, \mathcal{I}(1))$  is empty, and so we have:

$$0 \rightarrow H^0(P^3, \mathcal{O}_{P^3}(1)) \rightarrow H^0(C, \mathcal{O}_C(1)) \rightarrow 0$$

that is, the two are isomorphic. ■

**LEMMA 4** *The expression*

$$\frac{\mathcal{Q}}{\mathcal{P}}$$

*where  $\mathcal{Q}, \mathcal{P}$  are the polynomials, given in section 2.2 that vanish on all the singular points of the outline  $O$ , represents in coordinates an element of  $H^0(O, \mathcal{O}_O(1))$ , and hence an element of  $H^0(C, \mathcal{O}_C(1))$ , for  $C$  the contour generator.*

*Proof:* We have shown above that

$$\mathcal{P}u_3 + \mathcal{Q} = 0$$

on the contour generator; this is sufficient. ■

**Summary** Given the outline  $O$  of a surface, the space curve  $C$  given by applying the map

$$(x_0, x_1, x_2) \rightarrow (x_0\mathcal{P}, x_1\mathcal{P}, x_2\mathcal{P}, \mathcal{Q})$$

where  $\mathcal{P}$  and  $\mathcal{Q}$  are polynomials that can be determined by an overconstrained fitting process from the singularities of the outline, is the contour generator of the surface when it is viewed from the

point  $(0, 0, 0, 1)$ . Applying this map to a large number of points on the outline yields a set of points lying on the contour generator. As a result, the equations that vanish on the contour generator can be determined using a fitting process. Amongst this collection of equations lies the equation of a surface, projectively equivalent to the original surface.

#### 4. Obtaining the surface from the contour generator

The previous sections showed that it is possible to take the image outline of an algebraic surface and obtain a point in space and a space curve, which are respectively the focal point and the contour generator that gave rise to the outline, and are in the same coordinate frame - that is, the outline is obtained by projecting the reconstructed contour generator through the reconstructed focal point. The contour generator and focal point resulting from this reconstruction are projectively equivalent to the original contour generator and focal point.

Once the contour generator through a particular focal point is known, it is a relatively simple matter to obtain the surface, because of the strong relationship between the polynomials that vanish on the contour generator. There is one equation of degree  $d - 1$  that vanishes on the contour generator; if there were more, its degree would be  $(d - 1)^2$  or less. This equation is the first polar of the surface through  $(0, 0, 0, 1)$ ; the coefficients of this equation can be determined from a set of points on the contour generator by a fitting process. Call this equation  $T_m$ .

There is a five-dimensional linear space of equations of degree  $d$  that vanish on the contour generator, and this space can be determined by a fitting process. The equations lie in the linear space spanned by  $(u_0T, u_1T, u_2T, u_3T, S)$ . The fitting process will yield a basis for this space; call the elements of this basis  $(B_0, B_1, B_2, B_3, B_4)$ . Since

$$T_m = \frac{\partial S}{\partial u_3}$$

and  $S$  lies in the span of this basis, it follows that

$$S = \sum_{i=0}^{i=4} \nu_i B_i \quad (1)$$

for some set of constants  $\nu_i$  and that

$$T_m = \sum_{i=0}^{i=4} \nu_i \frac{\partial B_i}{\partial u_3} \quad (2)$$

Clearly, this equation is true in coefficients. The coefficients of  $T_m$  are known, as are the coefficients of  $B_i$ , and hence those of their partial derivatives. Since  $T_m$  must have at least 10 coefficients for the problem to be interesting ( $S$  must have degree 3 or greater for the result to be non-trivial), the terms  $\nu_i$  can be determined from equation 2. Once  $\nu_i$  are known,  $S$  can be reconstructed from equation 1. There must be at least one solution, because the curve is known to be a contour generator. In the general case, this is the only solution.

**LEMMA 5** *For  $S$  a generic surface viewed from a given focal point  $f$ , there is no other surface  $S'$ , such that the contour generator of  $S'$  viewed from  $f$  is the same curve as the contour generator of  $S$  viewed from  $f$ .*

*Proof:* The process that forms the contour generator is covariant. It is therefore sufficient to demonstrate that this lemma holds for a particular focal point. This focal point can be chosen to be  $(0, 0, 0, 1)$ .

If there are two different surfaces, whose equations are  $S$  and  $S'$ , which have the same contour generator when viewed through this focal point, the tangency relation that defines this contour generator must be the same for both surfaces, as the contour generator has degree  $d(d - 1)$ , and so only one form of degree  $d - 1$  can vanish on it. This can be written as:

$$\frac{\partial S'}{\partial u_3} = \lambda_0 \frac{\partial S}{\partial u_3}$$

where  $\lambda_0$  is an unknown constant to allow for scaling the equations (which does not affect the geometry of the underlying curve).

Furthermore, we have that the linear system of five degree  $d$  forms that vanishes on the contour generator is the same for each surface. Thus, in particular, there are constants  $\lambda_i$  such that

$$S' = \lambda_1 u_0 \frac{\partial S}{\partial u_3} + \lambda_2 u_1 \frac{\partial S}{\partial u_3} + \lambda_3 u_2 \frac{\partial S}{\partial u_3} + \lambda_4 u_3 \frac{\partial S}{\partial u_3} + \lambda_5 S$$

As a result, we can write:

$$\frac{\partial S'}{\partial u_3} = \lambda_1 u_0 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \lambda_2 u_1 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \lambda_3 u_2 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \lambda_4 u_3 \frac{\partial^2 S}{\partial u_3 \partial u_3} + (\lambda_5 + \lambda_4) \frac{\partial S}{\partial u_3}$$

This can be rewritten as:

$$\lambda_0 \frac{\partial S}{\partial u_3} = \lambda_1 u_0 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \lambda_2 u_1 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \lambda_3 u_2 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \lambda_4 u_3 \frac{\partial^2 S}{\partial u_3 \partial u_3} + (\lambda_5 + \lambda_4) \frac{\partial S}{\partial u_3}$$

By rearranging terms, and setting  $\mu_0 = \lambda_1$ ,  $\mu_1 = \lambda_2$ ,  $\mu_2 = \lambda_3$ ,  $\mu_3 = \lambda_4$ ,  $\mu_4 = \lambda_5 + \lambda_4 - \lambda_0$ , we obtain:

$$\mu_0 u_0 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \mu_1 u_1 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \mu_2 u_2 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \mu_3 u_3 \frac{\partial^2 S}{\partial u_3 \partial u_3} + \mu_4 \frac{\partial S}{\partial u_3} = 0$$

For the case that  $S' = \lambda_0 S$ , we must have that  $\lambda_1 = 0$ ,  $\lambda_2 = 0$ ,  $\lambda_3 = 0$ ,  $\lambda_4 = 0$ , and  $\lambda_5 = \lambda_0$ , so that all the  $\mu_i$  must vanish, and the equation is trivially true. If we have  $S'$ ,  $S$  where  $S' \neq \lambda_0 S$ , then there must be some solution for the above equation where not all  $\mu_i$  vanish. This yields an overdetermined system of equations *in the coefficients* of  $S$ , where the  $\mu_i$  are unknown. For these equations to be satisfied, the determinants of the coefficient matrices, which are easily shown to be non-trivial expressions in the coefficients of  $S$  alone, must vanish. In turn, these determinants represent constraints that the coefficients of  $S$  must satisfy, and so  $S$  is not a general surface. ■

## 5. Geometric ambiguities

The discussion above assumed abstract projection. Because the focal point for the reconstruction and the sections of the line bundle used to lift the outline were chosen arbitrarily, it is not surprising that the best possible reconstruction is

up to a projective transformation. However, this leaves a substantial ambiguity in the surface's geometry. It is often the case that the internal parameters of a camera are fully or partially known, and one might hope that a better reconstruction is possible in this case. Surprisingly, unless a modelbase is available, a better reconstruction appears impossible.

Consider a calibrated camera, where, without loss of generality, the focal point lies at  $(0, 0, 0, 1)$ . The outline of an object is formed by a cone of rays through this point, and tangent to the object itself. The intrinsic ambiguity of the reconstruction process must include all transformations of the object that fix this cone of rays and the focal point - this is the group of dilations of space, written as:

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{array}$$

where  $d \neq 0$ . If there is no modelbase, then the geometry of the surface observed must be given as a set of invariants to some transformation group. In particular, in most conceivable applications the description must be invariant to Euclidean transformations of space. It is easily verified<sup>6</sup> that the smallest subgroup of the projective group that contains both the Euclidean group and the dilations is the projective group itself. This means that to describe algebraic surfaces by invariants using only outline information and without reference to a modelbase, one must use projective invariants, whether the camera is calibrated or not.

However, a modelbase changes the ambiguity substantially. If, for example, there is a discrete modelbase with a small number of models, it is straightforward to extend the consistency approach of (Forsyth *et al.*, 1994) to yield a Euclidean reconstruction of a system of surfaces, though the study of ambiguities in the reconstruction appears to become difficult. It is not known whether these ambiguities allow reconstructions when there are parametrised families of models.

## 6. Discussion

There is now a constructive path from observations of outline points to the full projective geometry of the surface, which goes as follows:

- Fit an algebraic curve that is an outline to the observations - this process will also yield the degree of the surface (by a search over degrees  $d(d-1)$  for increasing  $d$ , if necessary).
- Determine the singularities of this fitted curve (using the coefficients of the curve).
- Compute the coefficients of the polynomials that would blow up these singularities, as described above in section 3.1.
- Use these coefficients to form a map from the plane to space (section 3.1). Apply this map to a large number of points on the outline, yielding a collection of points in space.
- Determine the unique surface of degree  $d-1$  passing through these points in space.
- Determine the five-parameter family of surfaces of degree  $d$  passing through these points in space.
- Determine the  $\nu_i$  of the previous section, using the methods given there.
- These  $\nu_i$  can be substituted into the equations above, to give the coefficients of the surface.

Although a simple implementation that successfully identifies cubic surfaces from synthetic outline data, exists, there are real difficulties in constructing an implementation of this approach that works in a practical vision system:

- Computing the outline from image data requires fitting high degree algebraic curves to edge points. The degree goes up as the square of the degree of the surface.
- Computing the contour generator from the outline is tricky, as it requires finding singularities in the outline. Unfortunately, a small change in the coefficients of the outline can lead to substantial errors in the computed singularities, both in location and in multiplicity. Such errors are guaranteed by the fact that we are using a fitted curve. Furthermore, the singularities must have special properties, for the curve to be an outline at all. This has advantages and disadvantages: the curve can be chosen from a smaller, more specialised class of

curves, which may make fitting more robust; at the same time, the curve produced by a general fitter cannot, in general, even be an outline.

- In practice, determining the surface from a system of points on the contour generator involves a process of fitting algebraic surfaces to points in space, and has the associated instabilities. Considerable precision in the points is required; in the experiments on synthetic data, this could be supplied, but it is doubtful whether such precision is available in practical situations.

Despite its present impracticality, this result is valuable, primarily because it shows that shape from outline is possible in the context of a very large and interesting range of surfaces, and thereby opens several promising avenues of research:

- *It is hard to be a contour generator.* In the case of algebraic surfaces, “most” curves are not contour generators, because either their degree, their genus, or the number and type of their singularities is wrong. There is good reason to believe that a similar result must hold for surfaces drawn from a “small” parametrised family of smooth surfaces, because the range of contour generators for a given surface is so small, although the mechanisms of proof and of computation may be more complex. This is the subject of active ongoing research.
- It is an example of a recognition algorithm that recognises an object drawn from a large, parametrised world (generic algebraic surfaces of degree three or greater) without searching a model-base. If this algorithm is presented with such a surface, it can (in principle) immediately describe the surface up to the intrinsic ambiguities of the viewing geometry. Given the way the algorithm is framed, verification appears to be either extremely difficult or impossible, and so the role of the model-base becomes uncertain.
- It suggests that the global properties of *systems* of contour generators are important objects of study. Compare the simple, neat structure of the family of contour generators on a projective algebraic surface with the extraordinary complexity of its aspect graph; in this case, the aspect graph is, in principle, redundant, because a single outline contains sufficient information to determine the entire surface. Fur-

thermore, in the case of projective algebraic surfaces, the system of contour generators is one case of a well understood class of objects: a linear system of curves on a surface. A study of this system might yield a much more practical algorithm for recognising a surface from two or three uncalibrated views, with an unknown transformation between the views, by exploiting the fact that each outline represents a curve drawn from a “small” (three-dimensional) linear system of curves.

- It opens a number of curious geometric questions; for example, what is the relationship between the six cusps on the outline of a cubic surface, and the surface? The contour generator is obtained from the outline by blowing up these six cusps on the outline; but, by standard results, if we were to extend this blowing up process to the whole plane, we would obtain a surface passing through the contour generator, and the degree of this surface could not be greater than three. Note that this is *not* a general cubic surface, because the six points blown up are not in general position, but appears to be a surface bearing some substantial relationship to the original cubic surface.

Determining a class of surfaces that is plastic enough to be useful for modelling a wide range of real objects, yet rigid enough to allow strong statements about the shape of a particular surface from a single outline, is the central issue in studying shape from contour. We have shown that algebraic surfaces represent one extreme; a very large class of surface that is so rigid that an outline determines a surface. There is room for much future work.

### Acknowledgements

Olivier De Barré, of the University of Iowa Mathematics Department pointed lemma 3 out to me, and showed me how it could be proven. This work has benefited from conversations with Tom Buchanan (who referred me to D’Almeida’s work), Olivier Faugeras, Margaret Fleck, Peter Giblin, Richard Hartley, Steve Maybank, Joe Mundy, John Oliensis, Charlie Rothwell, Richard Weiss and Andrew Zisserman. Anonymous referees provided extensive and extremely helpful comments.

This work was supported in part by a grant from United States Air Force Office of Scientific Research AFOSR-91-0361, in part by the National Science Foundation under award no. IRI-92-09729, in part by Magdalen College, Oxford, in part by the University of Iowa, and in part by a National Science Foundation Young Investigator Award with matching funds from GE, Tektronix, Rockwell and Eugene Rikel.

### Notes

1. A clear introduction to applying invariant theory in computer vision appears in (Mundy and Zisserman, 1992a).
2. Current systems using indexing functions have model-bases containing of the order of thirty objects.
3. There are a number of widely used terms for both curves, and no standard terminology has yet emerged.
4. Some algebraic curves in space must be given by more than two equations - see, for example (Ohm, 1980)
5. A generic element of a linear system is smooth away from its base points; see, say, Griffiths and Harris, 1986). Note that for a smooth surface, the system of contour generators has no base points, but if the surface is singular, the singularities of the surface are base points, and the contour generator may be singular.
6. The most practical technique is simply to form the commutators for the Lie algebra of the group containing both Euclidean transformations and dilations, as described in (Olver, 1986), and then note that the span of the set of commutators and generators is the Lie algebra of the projective group.

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